Pool Math

Anthony Lee Zhang

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1 Setup

You, A, are playing pool with someone better than you, R, for money. You must play an infinite number of games; you cannot exit. The rules are as follows. The loser of the last round picks a bet number, between \$0 and \$100. R can choose whether to make effort, or throw. If R makes an effort, she wins with probability $q > 0.5$. If R throws, she loses for sure.

2 Solving the model.

It's always optimal for R to bet \$100 if she loses. Now, suppose A commits to betting x if he loses. R can choose to throw or not throw.

Throwing: If R throws, wins always lead to losses, and losses lead to wins with probability q. The stationary probabilities p_W, p_L over win and loss states satisfies the flow equations:

$$p_W = p_L q$$

$$p_L = p_W + p_L (1 - q)$$

$$p_W + p_L = 1$$

Hence,

$$p_L (1 + q) = 1$$

$$p_W = \frac{q}{1+q}, p_L = \frac{1}{1+q}$$

When throwing, payoff is \$100 $(2q - 1)$ in the L state and $-x$ in the W state, hence,

$$\Pi_{throw} = \frac{q}{1+q} (-x) + \frac{1}{1+q} (100) (2q - 1)$$

Not throwing: If R doesn't throw, the stationary distributions are just R's win and loss probabilities:

$$p_W = q, p_L = 1 - q$$

Note that wins are more likely under the not throwing situation, $q > \frac{q}{1+q}$ When R isn't throwing, her expected payoff is \$100 $(2q-1)$ in the L state and $x(2q-1)$ in the W state, hence,

$$\Pi_{nothrow} = q(x)(2q-1) + (1-q)\,100\,(2q-1)$$

Optimal throwing. Hence the optimal throwing decision compares:

$$\Pi_{throw} > \Pi_{nothrow}$$

$$\frac{q}{1+q}(-x) + \frac{1}{1+q}(100)(2q-1) > q(x)(2q-1) + (1-q)\,100\,(2q-1)$$

As x increases, the cost of throwing increases and the gains from not throwing increase, that is,

$$\Pi'_{throw}(x) < 0, \Pi'_{nothrow}(x) > 0$$

In fact we can write:

$$\frac{1}{1+q}(100)(2q-1) - (1-q)\,100\,(2q-1) \geqslant q(x)(2q-1) - \frac{q}{1+q}(-x)$$

$$\left(\frac{1}{1+q} - (1-q)\right)(100)(2q-1) \geqslant xq\left((2q-1) + \frac{1}{1+q}\right)$$

$$\left(\frac{1-(1-q)(1+q)}{1+q}\right)(100)(2q-1) \geqslant xq\left(\frac{(2q-1)(q+1)+1}{1+q}\right)$$

$$\left(\frac{q^2}{1+q}\right)(100)(2q-1) \geqslant xq\left(\frac{2q^2-q+2q-1+1}{1+q}\right)$$

$$\left(\frac{q^2}{1+q}\right)(100)(2q-1) \geqslant xq\left(\frac{2q^2+q}{1+q}\right)$$

The LHS is the gains from throwing. This doesn't depend on x. It's the difference in probabilities of being in the L state, $\frac{1}{1+q}$ less $(1-q)$, times the payoff $(100)(2q-1)$. The RHS is the losses from throwing. It's the difference between expected gain of $qx(2q-1)$ in W states if not throwing, versus losing $\frac{q}{1+q}x$ in W states if throwing. Throwing is optimal if the gains are greater than the losses.

Nash equilibrium. The break-even x is:

$$x^* = \frac{\left(\frac{q^2}{1+q}\right)(100)(2q-1)}{q\left(\frac{2q^2+q}{1+q}\right)}$$

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$$x^* = \frac{q(100)(2q-1)}{2q^2+q}$$

$$x^* = \frac{q(100)(2q-1)}{q(2q+1)}$$

$$x^* = 100\frac{2q-1}{2q+1}$$

As $q \to 0.5$, the breakeven x goes to 0, but there is always a breakeven; i.e. if the cost of throwing is low enough throwing will always be optimal. As $q \to 1$, the breakeven goes to

$$\frac{100}{3}$$

By minimax theorem for zero-sum games, x^* is the unique Bayes-Nash equilibrium of the game.