An Online Appendix to "Depreciating Licenses"

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This is the online appendix for Weyl and Zhang (2018).

1 Proof of dynamic equilibrium uniqueness

1.1 Definitions

In Appendix Section A.4 of the paper, we defined the *pseudo-Bellman operator* $\mathcal{T}$ as:

$$\mathcal{T}\left[\hat{V}(\cdot)\right](\gamma) \equiv \max_{q}(q-\tau)\, p_{\hat{V}(\cdot),F(\cdot)}(q) + (1-q)\left[\gamma + \delta \mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}(\gamma') \mid \gamma\right]\right] \tag{1}$$

We claimed that fixed points of $\mathcal{T}$ correspond to equilibria of the dynamic Harberger license game. We also proved the $\mathcal{T}$ *net trade property*:

Claim 2. ($\mathcal{T}$ net trade property) Suppose that $\hat{V}(\cdot)$ is strictly increasing. Then $q^*_{\mathcal{T}}\left(\gamma;\ \hat{V}(\cdot)\right)$ satisfies:

- If $\tau = 1 - F(\gamma)$, we have $q^*_{\mathcal{T}}\left(\gamma;\ \hat{V}(\cdot)\right) = \tau$ and $p_{\hat{V}(\cdot),F(\cdot)}\left(q^*_{\mathcal{T}}\left(\gamma;\ \hat{V}(\cdot)\right)\right) = \gamma + \mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}(\gamma') \mid \gamma\right]$
- If $\tau < 1 - F(\gamma)$, we have $q^*_{\mathcal{T}}\left(\gamma;\ \hat{V}(\cdot)\right) \geqslant \tau$ and $p_{\hat{V}(\cdot),F(\cdot)}\left(q^*_{\mathcal{T}}\left(\gamma;\ \hat{V}(\cdot)\right)\right) \geqslant \gamma + \mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}(\gamma') \mid \gamma\right]$
- If $\tau > 1 - F(\gamma)$, we have $q^*_{\mathcal{T}}\left(\gamma;\ \hat{V}(\cdot)\right) \leqslant \tau$ and $p_{\hat{V}(\cdot),F(\cdot)}\left(q^*_{\mathcal{T}}\left(\gamma;\ \hat{V}(\cdot)\right)\right) \leqslant \gamma + \mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}(\gamma') \mid \gamma\right]$

We will use both of these in the following uniqueness proof.

1.2 Proof of equilibrium uniqueness

In the main text, we showed that equilibria of the dynamic Harberger license game exist and satisfy the net trade property. In this appendix, we show in addition that under an additional assumption, we can prove uniqueness of equilibrium.

Assumption 4. $\gamma_{t+1} \leqslant \gamma_t$ *with probability 1*

The proof proceeds as follows. In Claim 4 we will show that the pseudo-Bellman operator $\mathcal{T}$ is a contraction mapping for any seller types γ for which $\gamma < F^{-1}(1-\tau)$; that is, for all sellers with values below the $1-\tau$'th buyer quantile. In Claim 3 we show that the value function for these seller types can be solved for without reference to the value function above $F^{-1}(1-\tau)$. Thus, Claims 3 and 4 show that $\mathcal{T}$ uniquely pins down $V(\cdot)$ on $\gamma \in \left[0, F^{-1}(1-\tau)\right]$. Then, in Claim 5, we show that for any type $\tilde{\gamma} \geqslant F^{-1}(1-\tau)$, the derivative $V'(\tilde{\gamma})$ can be calculated using only knowledge of $V(\cdot)$ on values $\gamma \in [0, \tilde{\gamma}]$ lower than $\tilde{\gamma}$. Thus, once we know $V(\cdot)$ on the interval $\gamma \in \left[0, F^{-1}(1-\tau)\right]$, we can integrate $V'(\tilde{\gamma})$ upwards from $F^{-1}(1-\tau)$ to recover the entire unique equilibrium $V(\cdot)$ function.

Consider the distribution of entering buyer values $F(\gamma)$. We will define ρ-*quantile truncations* of $F(\cdot)$ as follows:

Definition. $\tilde{F}(\gamma;\ \rho)$ is the $\rho-$*quantile truncation* of $F(\gamma)$, defined as:

$$\tilde{F}(\gamma;\ \rho) = \begin{cases} F(\gamma) & F(\gamma) \leqslant \rho \\ 1 & F(\gamma) > \rho \end{cases}$$

In words, $\tilde{F}(\gamma;\ \rho)$ takes all probability mass above the ρth quantile of F, and puts it on the ρth quantile. In the two-stage Harberger license game, sellers at quantiles below $1-\tau$ set quantile markups between their quantile $1-F^{-1}(\gamma)$ and $1-\tau$. Intuitively, then, the demand distribution at quantiles above $1-\tau$ should not affect the behavior of these sellers; in particular, we can place all probability mass above buyer quantile $1-\tau$ at the $(1-\tau)$'th quantile, and this will not affect the behavior of sellers below the $(1-\tau)$'th quantile. This is formalized in the following claim.

Claim 3. (Truncation property) Suppose that $V(\cdot)$ is a stationary equilibrium value function for the Harberger license game under tax τ, with entering buyer distribution F. Then, $V(\cdot)$ restricted to the interval $\gamma \in \left[0,\ F^{-1}(\rho)\right]$ is a stationary equilibrium value function for the Harberger license game under tax τ, with entering buyer distribution $\tilde{F}(\gamma;\ \rho)$, for any $\rho \geqslant 1-\tau$.

Proof. An equilibrium of $V(\cdot)$ is a fixed point of the pseudo-Bellman operator $\mathcal{T}$, that is, it satisfies, for all γ:

$$V(\gamma) = \max_q (q-\tau)\, p_{V(\cdot),F(\cdot)}(q) + (1-q)\left[\gamma + \delta \mathbb{E}_{G(\cdot|\cdot)}\left[V\left(\gamma'\right) \mid \gamma\right]\right].$$

We want to show that, for any such V, we also have, for any $\rho \geqslant 1-\tau$,

$$V(\gamma) = \max_q (q-\tau)\, p_{V(\cdot),\tilde{F}(\gamma;\ \rho)}(q) + (1-q)\left[\gamma + \delta \mathbb{E}_{G(\cdot|\cdot)}\left[V\left(\gamma'\right) \mid \gamma\right]\right] \ \forall \gamma \leqslant F^{-1}(\rho).$$

In other words, we want to show that the truncation of F and $V(\cdot)$ does not affect the optimization problem of any type with $\gamma < F^{-1}(\rho)$.

Recall the definition of the WTP function:

$$WTP(\gamma) \equiv \gamma + \delta \mathbb{E}_{G(\cdot|\cdot)} \left[V(\gamma') \mid \gamma \right].$$

We have from Assumption 4 that $G(\gamma' \mid \gamma)$ satisfies $\gamma' \leqslant \gamma$ with probability 1, so evaluating the WTP function at γ only requires evaluating V on the interval $[0, \gamma]$. Thus, for all $\gamma \in \left[0, F^{-1}(\rho)\right]$, we can still evaluate WTP using V truncated to the interval $\left[0, F^{-1}(\rho)\right]$. Under $\tilde{F}(\gamma;\ \rho)$, the inverse demand function becomes:

$$\tilde{p}(q;\ \rho) = \begin{cases} p(q) & 1 - q \leqslant \rho \\ p(1-\rho) & 1 - q > \rho \end{cases}$$

Thus, by construction, the modified inverse demand function agrees with p on the inverval $[0, \rho]$, that is:

$$\tilde{p}(q;\ \rho) = p(q)\ \forall \{q:\ 1 - q \in [0, \rho]\}.$$

From Claim 2, under any increasing candidate $\hat{V}$ function, any seller with quantile $F^{-1}(\gamma) \in [0,\, 1-\tau]$ chooses some $1 - q \in [F(\gamma),\, 1-\tau]$. It must then be that the behavior of the inverse demand function $p(q)$ outside the range $1 - q \in \left[F^{-1}(\gamma),\, 1-\tau\right]$ does not affect these sellers' optimization problem (as long as $p(q)$ is derived from an increasing candidate $\hat{V}$ function, i.e. is monotone). Likewise, from Claim 2, any seller quantile $F(\gamma) \in [0,\, 1]$ chooses some $1 - q \in [1-\tau,\, F(\gamma)]$, hence the behavior of $p(q)$ outside the range $1 - q \in [1-\tau,\, F(\gamma)]$ does not affect the optimization problem of seller value $F^{-1}(\gamma)$.

In the ρ-truncated problem, sellers with values $F(\gamma) \leqslant 1-\tau$ care about $p(q)$ in the range $[0, 1-\tau]$, and sellers with values $1-\tau \leqslant F(\gamma) \leqslant \rho$ care about $p(q)$ in the range $[1-\tau, \rho]$. Since $\tilde{p}(q;\ \rho) = p(q)$ on the interval $[0, \rho]$, $\tilde{p}(q;\ \rho)$ is identical to $p(q)$ from the perspective of all sellers types with quantiles $F^{-1}(\gamma) \in [0, \rho]$. Hence there is no seller type in the ρ-truncated problem whose optimization problem is affected by the truncation of $p(q)$. Thus, any optimal policy $q^*(\gamma)$ and value function $V(\gamma)$ in the original problem remains optimal in the truncated problem, proving the claim. □

We will now consider the most extreme possible truncation, $\rho = 1-\tau$. Under this truncation, there are no types with values strictly above the $(1-\tau)$th quantile; thus, all seller types in the truncated interval are net sellers. In the following claim, we use the net seller property to show that $\mathcal{T}$ is a contraction mapping on the $(1-\tau)$-truncated problem.

Claim 4. (Contraction property) For any $\tau, F(\cdot)$, consider the $(1-\tau)$-truncated problem, with entering buyer distribution $\tilde{F}(\gamma,\, 1-\tau)$. $\mathcal{T}$ is a contraction mapping on this problem, hence admits a unique fixed point. Moreover, the unique fixed point $V(\cdot)$ of $\mathcal{T}$ must be continuous.

Proof. Once again, $\mathcal{T}$ is:

$$\mathcal{T}\left[\hat{V}\right](\gamma) = \max_{q}(q-\tau)\, p_{\hat{V}(\cdot),F(\cdot)}(q) + (1-q)\left[\gamma + \delta\mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(\gamma'\right) \mid \gamma\right]\right].$$

Consider $\hat{V}, \tilde{V}$ s.t. $\sup_{\gamma}\left|\hat{V}(\gamma) - \tilde{V}(\gamma)\right| \leqslant a$. We want to bound the sup norm difference between $\mathcal{T}\left[\hat{V}\right](\gamma)$ and $\mathcal{T}\left[\tilde{V}\right](\gamma)$. First, note that from the definition of $p_{\hat{V}(\cdot),F(\cdot)}(\cdot)$,

$$p_{\hat{V}(\cdot),F(\cdot)}(q) \equiv \left\{p:\ P_{\hat{V}(\cdot),F(\cdot)}\left[\gamma + \delta\mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(\gamma'\right) \mid \gamma\right] > p\right] = q\right\},$$

hence we have that

$$\left|p_{\hat{V}(\cdot),F(\cdot)}(q) - p_{\tilde{V}(\cdot),F(\cdot)}(q)\right| \leqslant \left|\delta\mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(\gamma'\right) \mid \gamma\right] - \delta\mathbb{E}_{G(\cdot|\cdot)}\left[\tilde{V}\left(\gamma'\right) \mid \gamma\right]\right| \leqslant \delta a.$$

Now, writing $\mathcal{T}\left[\hat{V}\right]$:

$$\mathcal{T}\left[\hat{V}\right](\gamma) = \left[q^*_{\mathcal{T}}\left(\gamma; \hat{V}\right) - \tau\right] p_{\hat{V}(\cdot),F(\cdot)}\left(q^*_{\mathcal{T}}\left(\gamma; \hat{V}\right)\right) + \left(1 - q^*_{\mathcal{T}}\left(\gamma; \hat{V}\right)\right)\left[\gamma + \delta\mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(\gamma'\right) \mid \gamma\right]\right].$$

We will show that, if under $\tilde{V}$ we fix the sale probability at $q^*_{\mathcal{T}}\left(\gamma; \hat{V}\right)$, we lose at most δa. Hence the sup norm difference between $T\left[\hat{V}\right]$, $T\left[\tilde{V}\right]$ is at most δa.

We can separately deal with the "buyer" inverse demand and "seller" continuation value terms. For the seller's continuation value term, note that:

$$\gamma + \delta\mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(\gamma'\right) \mid \gamma\right] \leqslant \gamma + \delta\mathbb{E}_{G(\cdot|\cdot)}\left[\tilde{V}\left(\gamma'\right) \mid \gamma\right] + \delta a.$$

Hence,

$$\begin{aligned}\left(1 - q^*_{\mathcal{T}}\left(\gamma; \hat{V}\right)\right)\left[\gamma + \delta\mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(\gamma'\right) \mid \gamma\right]\right] \leqslant \\ \left(1 - q^*_{\mathcal{T}}\left(\gamma; \hat{V}\right)\right)\left[\gamma + \delta\mathbb{E}_{G(\cdot|\cdot)}\left[\tilde{V}\left(\gamma'\right) \mid \gamma\right]\right] + \left(1 - q^*_{\mathcal{T}}\left(\gamma; \hat{V}\right)\right)\delta a\end{aligned}$$

For the buyer inverse demand term,

$$\begin{aligned}\left[q^*_{\mathcal{T}}\left(\gamma; \hat{V}\right) - \tau\right] p_{\hat{V}(\cdot),F(\cdot)}\left(q^*_{\mathcal{T}}\left(\gamma; \hat{V}\right)\right) \leqslant \\ \left[q^*_{\mathcal{T}}\left(\gamma; \hat{V}\right) - \tau\right] p_{\tilde{V}(\cdot),F(\cdot)}\left(q^*_{\mathcal{T}}\left(\gamma; \hat{V}\right)\right) + \left|q^*_{\mathcal{T}}\left(\gamma; \hat{V}\right) - \tau\right|\delta a\end{aligned}$$

Adding these inequalities, we have that:

$$\begin{aligned}&\mathcal{T}\left[\hat{V}\right](\gamma) = \\ &\quad \left[q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)-\tau\right] p_{\hat{V}(\cdot),F(\cdot)}\left(q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)\right) + \left(1-q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)\right)\left[\gamma+\delta\mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(\gamma'\right)\mid\gamma\right]\right] \leqslant \\ &\qquad \mathcal{T}\left[\tilde{V}\right](\gamma) + \left(1-q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)\right)\delta a + \left|q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)-\tau\right|\delta a.\end{aligned}$$

By Claim 2, we know that all sellers in the truncated range are net sellers, that is, $1-q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)\leqslant 1-\tau$, or $q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)\geqslant\tau$. Hence $\left|q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)-\tau\right| < \left|q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)\right|$, hence we have:

$$\left(1-q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)\right)\delta a + \left|q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)-\tau\right|\delta a \leqslant \left|\left(1-q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)\right)\right|\delta a + \left|q^*_{\mathcal{T}}\left(\gamma;\hat{V}\right)\right|\delta a \leqslant \delta a. \quad (2)$$

We have thus shown that, for all γ,

$$\mathcal{T}\left[\hat{V}\right](\gamma) \leqslant \mathcal{T}\left[\tilde{V}\right](\gamma) + \delta a.$$

$\hat{V}$ and $\tilde{V}$ were arbitrary, so by switching their roles we get:

$$\mathcal{T}\left[\tilde{V}\right](\gamma) \leqslant \mathcal{T}\left[\hat{V}\right](\gamma) + \delta a$$

$$\implies \sup_{\gamma}\left[\mathcal{T}\left[\tilde{V}\right](\gamma) - \mathcal{T}\left[\hat{V}\right](\gamma)\right] \leqslant \delta a.$$

Hence $\mathcal{T}$ is a contraction mapping of modulus δ.

To show that the unique fixed point $V(\cdot)$ must be continuous, we will show that for an increasing but possibly discontinuous candidate value function $\hat{V}(\cdot)$, $\mathcal{T}\left[\hat{V}\right]$ must be continuous. Once again, $\mathcal{T}$ is:

$$\mathcal{T}\left[\hat{V}\right] = \max_{q}\,(q-\tau)\,p_{\hat{V}(\cdot),F(\cdot)}(q) + (1-q)\left[\gamma+\delta\mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(\gamma'\right)\mid\gamma\right]\right].$$

We need only to show that $\mathbb{E}_{G(\cdot|\cdot)}\left[V\left(\gamma'\right)\mid\gamma\right]$ is continuous in γ, since all other components of the maximand are continuous in γ. Since $\hat{V}$ is strictly increasing, the generalized inverse function $\hat{\Gamma}(\nu)\equiv\arg\min_{\gamma}|V(\gamma)-\nu|$ is everywhere well-defined. Since $\hat{V}(\cdot)$ is bounded, the "layer cake" representation of its expected value obtains. Letting $\bar{V}\equiv\max_{\gamma}\hat{V}(\gamma)$, we have:[1]

$$\mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(\gamma'\right)\mid\gamma\right] = \int_0^{\bar{V}} 1-G\left(\hat{\Gamma}(\nu)\mid\gamma\right)d\nu.$$

Since $G\left(\gamma'\mid\gamma\right)$ is continuous in γ for any γ', the integral $\int_0^{\bar{V}} G\left(\hat{\Gamma}(\nu)\mid\gamma\right)d\nu$ is also continuous in

[1]This definition of the layer-cake integral is slightly wrong, failing if $\hat{V}(\cdot)$ and $G(\cdot)$ have discontinuities at the same value of γ. This can be fixed by redefining G such that that probability mass falls at the correct side of each $\hat{V}$ discontinuity.

γ.[2] $\square$

Remark. Equation 2 is the step where the contraction property fails in general, and is why we need this truncation argument. Suppose for example $\tau = 1$, so that $\tau > q$. Then the continuation value term has modulus $(1-q)\,\delta a$ and the buyer price term has modulus $|q-\tau|\,\delta a = |1-q|\,\delta a$. So the total modulus bound is $2\,(1-q)\,\delta a$, and we can't guarantee that $\mathcal{T}$ is a contraction.

Claim 4 shows that the equilibrium value function $V(\cdot)$ is uniquely pinned down in the $(1-\tau)$-truncated problem, and Claim 3 shows that the equilibrium $V(\cdot)$ functions from the original and truncated problems must agree. Hence we have shown that the equilibrium $V(\cdot)$ is unique at least in the truncated interval $\left[0,\ F^{-1}(1-\tau)\right]$. In Claim 5, we show that, in any ρ-truncated equilibrium, we can calculate the derivative of the value function $\frac{dV}{d\gamma}\,|_{\gamma=F^{-1}(\rho)}$ at the boundary type $F^{-1}(\rho)$.

Claim 5. (Envelope theorem) The envelope theorem applies to any equilibrium:

$$\frac{dV}{d\gamma} = \frac{\partial}{\partial\gamma}\left[\left(q^*(\gamma)-\tau\right)p_{V(\cdot),F(\cdot)}\left(q^*(\gamma)\right) + \left(1-q^*(\gamma)\right)\left[\gamma + \delta\mathbb{E}_{G(\cdot|\cdot)}\left[V\left(\gamma'\right)\mid\gamma\right]\right]\right]$$

$$= \left(1-q^*(\gamma)\right)\left[1+\delta\frac{\partial\mathbb{E}_{G(\cdot|\cdot)}\left[V\left(\gamma'\right)\mid\gamma\right]}{\partial\gamma}\right]. \tag{3}$$

Proof. Following Milgrom and Segal (2002), we need to show that the conjectured derivative:

$$(1-q)\left[1+\delta\frac{\partial\mathbb{E}_{G(\cdot|\cdot)}\left[V\left(\gamma'\right)\mid\gamma\right]}{\partial\gamma}\right]$$

is finite for any choice of q. Using the layer cake representation once again:

$$\mathbb{E}_{G(\cdot|\cdot)}\left[\hat{V}\left(\gamma'\right)\mid\gamma\right] = \int_0^{\bar{V}} 1 - G\left(\hat{\Gamma}(\nu)\mid\gamma\right)d\nu.$$

Leibniz' formula implies that

$$\frac{\partial\mathbb{E}_{G(\cdot|\cdot)}\left[V\left(\gamma'\right)\mid\gamma\right]}{\partial\gamma} = -\int_0^{\bar{V}}\frac{\partial G\left(\hat{\Gamma}(\nu)\mid\gamma\right)}{\partial\gamma}d\nu.$$

We have from Assumption 3 that $\frac{\partial G\left(\hat{\Gamma}(\nu)|\gamma\right)}{\partial\gamma}$ exists for any ν, hence this quantity is finite for any q, and thus the envelope theorem applies. $\square$

Thus, in any ρ-truncated equilibrium, we can evaluate $q^*(\gamma)$ for the boundary type $\gamma = F^{-1}(\rho)$. Moreover, since Assumption 4 the transition probability distribution G satisfies that

[2] This follows even without assuming differentiability of G, from a "set excision" argument: for any ϵ, there is a δ satisfying continuity for $G\left(\hat{\Gamma}(\nu)\mid\gamma\right)$ except on a set of arbitrarily small ν-measure, and G is bounded between $[0,1]$ on the excised set.

$\gamma_{t+1} < \gamma_t$ a.s., the expectation $\mathbb{E}_{G(\cdot|\cdot)}[V(\gamma') \mid \gamma]$ puts positive probability only on values $\gamma' < \gamma$. Hence, in any ρ-truncated equilibrium, we can evaluate the derivative $\frac{dV}{d\gamma}$ for the boundary type $\gamma = F^{-1}(\rho)$ using only knowledge of the equilibrium V on the truncated interval $\left[0, F^{-1}(\rho)\right]$. Thus, after solving for V on the interval $\left[0, F^{-1}(1-\tau)\right]$ using the contraction mapping of Claim 4, we can integrate the envelope derivative formula in (3) to recover the rest of the equilibrium $V(\cdot)$ function:

$$V(\gamma) = \int_{F^{-1}(1-\tau)}^{F^{-1}(\gamma)} (1 - q^*(\tilde{\gamma})) \left[1 + \delta \frac{\partial \mathbb{E}_{G(\tilde{\gamma}'|\tilde{\gamma})}[V(\tilde{\gamma}') \mid \tilde{\gamma}]}{\partial \tilde{\gamma}}\right] d\tilde{\gamma} \; \forall \gamma > F^{-1}(1-\tau).$$

We have thus proved that, for any τ, $F(\gamma)$, $G(\gamma' \mid \gamma)$ satisfying our assumptions, there is a unique equilibrium of the dynamic Harberger license game.

Remark. We used Assumption 4 at two points. First, the proof of Claim 3 requires that the $\mathcal{T}$ operator applied to the truncated value distribution does not reference values of γ above the truncation quantile. Second, to evaluate the envelope derivative formula in Claim 5, we once again need to be able to reference only values of γ below the truncation quantile.

References

Milgrom, Paul, and Ilya Segal. 2002. "Envelope theorems for arbitrary choice sets." *Econometrica*, 70(2): 583–601.

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