Surplus Monotonicity in Second-Best Bargaining

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1 Introduction

Consider the Myerson and Satterthwaite [1] (henceforth MS) second-best optimization problem. Fix buyer and seller value distributions F_B, F_S satisfying marginal revenue/cost regularity. For simplicity suppose F_B, F_S are commonly supported on $[0, 1]$. The optimization problem for second-best welfare, which we will call $SBW(F_B, F_S)$, is:

$$SBW(F_B, F_S) = \max_{x(v_B, v_S)} \int_0^1 \int_0^1 (v_B - v_S)\, x(v_B, v_S)\, dF_B(v_B)\, dF_S(v_S)$$

$$\text{s.t.} \int_0^1 \int_0^1 \left(v_B - \frac{1 - F_B(v_B)}{f_B(v_B)}\right) - \left(v_S + \frac{F_S(v_S)}{f_S(v_S)}\right) x(v_B, v_S)\, dF_B(v_B)\, dF_S(v_S) \geqslant 0$$

Note that, in the unconstrained problem of maximizing first-best trade, it is clear that optimal welfare is monotone under first-order stochastic dominance shifts of the distributions of buyer and seller values. It is not immediately clear that this monotonicity also applies to the second-best problem; the issue is that it is not immediately clear whether the revenue constraint behaves well under stochastic dominance shifts.

In this note I prove Claim 1, that second-best welfare increases when we FOSD-increase the distribution F_B. Note that a similar statement for lowering F_S follows essentially symmetrically – we will for simplicity only analyze the buyer case.

Claim 1. $\tilde{F}_B \geqslant_{FOSD} F_B$ implies that $SBW\left(\tilde{F}_B, F_S\right) \geqslant_{FOSD} SBW(F_B, F_S)$

2 Intuition

We will prove Claim 1 by taking the optimal *quantile trade function* associated with F_B, and showing that it satisfies constraints and achieves weakly higher welfare under $\tilde{F}_B$. In Subsection 3.1, I show that weakly higher welfare is achieved, which is straightforwards. I show constraint satisfaction in 3.2. This is more difficult; it is not immediately clear why the quantile trade function should behave well with respect to the constraints, since the constraints involve marginal revenues and costs which are relatively complex nonlinear functions of value distributions. However, while marginal revenues and marginal costs are not necessarily monotone with respect to FOSD-shifts of the distributions F_S and F_B, certain integrals of MR's and MC's are. With a few integration-by-parts tricks, the constraints for the second-best optimization problem can be written only in terms of these integrals, and we use this to demonstrate constraint satisfaction.

3 Proof

We will prove Claim 1 by taking the optimal $x(v_B, v_S)$ function under F_B, F_S and using it to construct a candidate $x(v_B, v_S)$ function for $\tilde{F}_B, F_S$ which is feasible, and attains weakly higher welfare. We will refer to the optimal $x(\cdot, \cdot)$ function under F_B, F_S as $x^*(\cdot, \cdot)$.

Define the inverse quantile functions

$$\tilde{\boldsymbol{v}}_B(q_B) = \tilde{F}_B^{-1}(q_B), \ \boldsymbol{v}_B(q_B) = F_B^{-1}(q_B), \ \boldsymbol{v}_S(q_S) = F_S^{-1}(q_S)$$

We construct a candidate $\tilde{\boldsymbol{x}}(\cdot,\cdot)$ function for $\tilde{F}_B, F_S$ as follows:

Definition 1. Define $\tilde{\boldsymbol{x}}(\cdot,\cdot)$ as:

$$\tilde{\boldsymbol{x}}(v_B, v_S) = \boldsymbol{x}^*\left(\boldsymbol{v}_B\left(\tilde{F}_B(v_B)\right), v_S\right)$$

in words, $\tilde{\boldsymbol{x}}(\cdot,\cdot)$ and $\boldsymbol{x}^*(\cdot,\cdot)$ are the same in quantile space; that is, $\tilde{\boldsymbol{x}}(\tilde{v}_B, v_S)$ is equal to $\boldsymbol{x}^*(v_B, v_S)$ evaluated at v_B equal to the F-quantile equal to the $\tilde{F}(\tilde{v})$. Or, in other words, if the 50th percentile buyer and seller trade under $\boldsymbol{x}^*(\cdot,\cdot)$, they also trade under $\tilde{\boldsymbol{x}}(\tilde{v}_B, v_S)$.

We will prove Claim 1 by showing that:

Claim 2. Under distributions $\tilde{F}_B, F_S$,

1. The constructed $\tilde{\boldsymbol{x}}(v_B, v_S)$ attains weakly higher welfare than $SBW(F_B, F_S)$.
2. The constructed $\tilde{\boldsymbol{x}}(v_B, v_S)$ attains nonnegative expected revenue.

We separately prove 1. and 2.

3.1 Proof of 1

Note that we can specify both $\boldsymbol{x}^*(\cdot,\cdot)$ and $\tilde{\boldsymbol{x}}(\cdot,\cdot)$ in quantile space; that is, define

$$\boldsymbol{y}(q_B, q_S) = \boldsymbol{x}^*(\boldsymbol{v}_B(q_B), \boldsymbol{v}_S(q_S))$$

Note that, by construction of $\tilde{\boldsymbol{x}}(\cdot,\cdot)$, we also have that

$$\boldsymbol{y}(q_B, q_S) = \tilde{\boldsymbol{x}}(\tilde{\boldsymbol{v}}_B(q_B), \boldsymbol{v}_S(q_S))$$

Now consider the objective function of the optimization problem, which we will call:

$$SW(\boldsymbol{x}(\cdot,\cdot), F_B(\cdot), F_S(\cdot)) = \int_0^1 \int_0^1 (v_B - v_S)\, \boldsymbol{x}(v_B, v_S)\, dF_B(v_B)\, dF_S(v_S)$$

We can change variables to quantile space, writing:

$$SW(\boldsymbol{x}(\cdot,\cdot), F_B(\cdot), F_S(\cdot)) = \int_0^1 \int_0^1 (\boldsymbol{v}_B(q_B) - \boldsymbol{v}_S(q_S))\, \boldsymbol{y}(q_B, q_S)\, dq_B\, dq_S$$

Likewise,

$$SW\left(\tilde{\boldsymbol{x}}(\cdot,\cdot), \tilde{F}_B(\cdot), F_S(\cdot)\right) = \int_0^1 \int_0^1 (\tilde{\boldsymbol{v}}_B(q_B) - \boldsymbol{v}_S(q_S))\, \boldsymbol{y}(q_B, q_S)\, dq_B\, dq_S$$

By definition, $\tilde{F}_B \geqslant_{FOSD} F_B$ implies that all quantiles of $\tilde{F}_B$ are higher than those of F_B, that is, $\tilde{\boldsymbol{v}}_B(q_B) \geqslant \boldsymbol{v}_B(q_B)$ for all q_B. Hence,

$$\tilde{\boldsymbol{v}}_B(q_B) - \boldsymbol{v}_S(q_S) \geqslant \boldsymbol{v}_B(q_B) - \boldsymbol{v}_S(q_S) \ \forall q_B, q_S$$

Hence, since all other terms in the integral are identical, we have that

$$SW\left(\tilde{\boldsymbol{x}}(\cdot,\cdot), \tilde{F}_B(\cdot), F_S(\cdot)\right) \geqslant SW(\boldsymbol{x}(\cdot,\cdot), F_B(\cdot), F_S(\cdot))$$

proving part 1 of Claim 2.

3.2 Proof of 2

We write total revenue as:

$$Rev\left(\mathbf{x}\left(\cdot,\cdot\right),F_B,F_S\right)=\int_0^1\int_0^1\left(v_B-\frac{1-F_B\left(v_B\right)}{f_B\left(v_B\right)}\right)-\left(v_S+\frac{F_S\left(v_S\right)}{f_S\left(v_S\right)}\right)\mathbf{x}\left(v_B,v_S\right)dF_B\left(v_B\right)dF_S\left(v_S\right)$$

And, we need $Rev\left(\mathbf{x}\left(\cdot,\cdot\right),F_B,F_S\right)\geqslant 0$. First, note that we can split this into the buyer and seller pieces:

$$\begin{aligned}Rev\left(\mathbf{x}\left(\cdot,\cdot\right),F_B,F_S\right)=&\int_0^1\int_0^1\left(v_B-\frac{1-F_B\left(v_B\right)}{f_B\left(v_B\right)}\right)\mathbf{x}\left(v_B,v_S\right)dF_B\left(v_B\right)dF_S\left(v_S\right)\\&-\int_0^1\int_0^1\left(v_S+\frac{F_S\left(v_S\right)}{f_S\left(v_S\right)}\right)\mathbf{x}\left(v_B,v_S\right)dF_B\left(v_B\right)dF_S\left(v_S\right)\end{aligned}$$

For the seller piece, we can write:

$$\begin{aligned}\int_0^1\int_0^1\left(v_S+\frac{F_S\left(v_S\right)}{f_S\left(v_S\right)}\right)\mathbf{x}^*\left(v_B,v_S\right)dF_B\left(v_B\right)dF_S\left(v_S\right)&=\int_0^1\int_0^1\left(\mathbf{v}_S\left(q_S\right)+\frac{F_S\left(\mathbf{v}_S\left(q_S\right)\right)}{f_S\left(\mathbf{v}_S\left(q_S\right)\right)}\right)\mathbf{y}\left(v_B,v_S\right)dq_B\,dq_S\\&=\int_0^1\int_0^1\left(v_S+\frac{F_S\left(v_S\right)}{f_S\left(v_S\right)}\right)\mathbf{y}\left(v_B,v_S\right)d\tilde{F}_B\left(v_B\right)dF_S\left(v_S\right)\end{aligned}$$

So, the revenue from sellers is the same under $\mathbf{x}^*\left(\cdot,\cdot\right)$ and $\tilde{\mathbf{x}}\left(\cdot,\cdot\right)$. In words, if we hold fixed the quantile trade function $\mathbf{y}\left(v_B,v_S\right)$ when moving from F_B to $\tilde{F}_B$, all seller types face the same marginal probability of trade, hence revenue from sellers is unchanged.

So, we only need to show that revenue from buyers is weakly greater under $\tilde{\mathbf{x}},\tilde{F}$, that is:

$$\int_0^1\int_0^1\left(v_B-\frac{1-\tilde{F}_B\left(v_B\right)}{\tilde{f}_B\left(v_B\right)}\right)\tilde{\mathbf{x}}\left(v_B,v_S\right)d\tilde{F}_B\left(v_B\right)dF_S\left(v_S\right)\geqslant\int_0^1\int_0^1\left(v_B-\frac{1-F_B\left(v_B\right)}{f_B\left(v_B\right)}\right)\mathbf{x}^*\left(v_B,v_S\right)dF_B\left(v_B\right)dF_S\left(v_S\right)\tag{1}$$

3.2.1 Buyer Revenue

For arbitrary implementable $\mathbf{x}\left(\cdot,\cdot\right)$, the buyer revenue is:

$$BRev\left(\mathbf{x}\left(\cdot,\cdot\right),F_B\left(\cdot\right)\right)=\int_0^1\int_0^1\left(v_B-\frac{1-F_B\left(v_B\right)}{f_B\left(v_B\right)}\right)\mathbf{x}\left(v_B,v_S\right)dF_B\left(v_B\right)dF_S\left(v_S\right)$$

In this notation, the constraint (1) can be written as:

$$BRev\left(\tilde{\mathbf{x}}\left(\cdot,\cdot\right),\tilde{F}_B\left(\cdot\right)\right)\geqslant BRev\left(\mathbf{x}^*\left(\cdot,\cdot\right),F_B\left(\cdot\right)\right)\tag{2}$$

We will rearrange this doing a few integrations by parts. First, since the seller term $dF_S\left(v_S\right)$ doesn't enter into the integrand, we integrate it out. Define

$$\mathbf{p}\left(v_B;\ \mathbf{x}\left(\cdot,\cdot\right)\right)=\int_0^1\mathbf{x}\left(v_B,v_S\right)dF_S\left(v_S\right)$$

In the remainder of this subsection, we will suppress the dependence of $\mathbf{p}\left(\cdot\right)$ on $\mathbf{x}\left(\cdot,\cdot\right)$, but this will be important later.

Implementability requires that $\mathbf{p}\left(\cdot\right)$ is nondecreasing. Hence, we have

$$BRev=\int_0^1\left(v_B-\frac{1-F_B\left(v_B\right)}{f_B\left(v_B\right)}\right)\left(\int_0^1\mathbf{x}\left(v_B,v_S\right)dF_B\left(v_B\right)\right)dF_S\left(v_S\right)=\int_0^1\left(v_B-\frac{1-F_B\left(v_B\right)}{f_B\left(v_B\right)}\right)\mathbf{p}\left(v_B\right)dF_B\left(v_B\right)$$

Now, we'll replace $\mathbf{p}(v_B)$ by $\int_0^{\mathbf{p}(v_B)} dp$, giving:

$$\mathrm{BRev} = \int_0^1 \int_0^{\mathbf{p}(v_B)} \left(v_B - \frac{1 - F_B(v_B)}{f_B(v_B)}\right) dp\, dF_B(v_B)$$

The benefit of writing this is that we can interchange the order of integration. Defining $\bar{v}_B(p) = \mathbf{p}^{-1}(v_B)$, this gives us:

$$\mathrm{BRev} = \int_0^1 \int_{\bar{v}_B(p)}^1 \left(v_B - \frac{1 - F_B(v_B)}{f_B(v_B)}\right) dF_B(v_B)\; dp$$

This is measure-theoretically legitimate assuming marginal revenue is bounded, and the bounds are correct because $\mathbf{p}(v_B)$ is an increasing function.

Now, we focus on the inner integral. For some p, consider:

$$\begin{aligned}
d\mathrm{BRev} &= \int_{\bar{v}_B(p)}^1 \left(v_B - \frac{1 - F_B(v_B)}{f_B(v_B)}\right) dF_B(v_B) \\
&= \int_{\bar{v}_B(p)}^1 \left(v_B - \frac{1 - F_B(v_B)}{f_B(v_B)}\right) f_B(v_B)\, dv_B \\
&= \int_{\bar{v}_B(p)}^1 v_B f_B(v_B)\, dv_B - (1 - F_B(v_B))\, dv_B
\end{aligned}$$

Now,

$$\int_{\bar{v}_B(p)}^1 v_B f_B(v_B)\, dv_B = E[v_B \mid v_B > \bar{v}_B(p)]\, P[v_B > \bar{v}_B(p)]$$

$$\int_{\bar{v}_B(p)}^1 (1 - F_B(v_B))\, dv_B = E[v_B - \bar{v}_B(p) \mid v_B > \bar{v}_B(p)]\, P[v_B > \bar{v}_B(p)]$$

Hence,

$$\begin{aligned}
\int_{\bar{v}_B(p)}^1 v_B f_B(v_B)\, dv_B - (1 - F_B(v_B))\, dv_B &= E[\bar{v}_B(p) \mid v_B > \bar{v}_B(p)]\, P[v_B > \bar{v}_B(p)] \\
&= \bar{v}_B(p)(1 - F_B(\bar{v}_B(p)))
\end{aligned}$$

In sum, we have shown that

$$\mathrm{BRev} = \int_0^1 d\mathrm{BRev}\, dp = \int_0^1 \bar{v}_B(p)(1 - F_B(\bar{v}_B(p)))\; dp \tag{3}$$

3.2.2 Properties of $\mathbf{p}(v_B;\, \mathbf{x}(\cdot,\cdot), F_B)$

Now, by our definition of $\mathbf{p}(v_B;\, \mathbf{x}^*(\cdot,\cdot))$, we have:

$$\begin{aligned}
\mathbf{p}(v_B;\, \mathbf{x}^*(\cdot,\cdot)) &= \int_0^1 \mathbf{x}^*(v_B, v_S)\, dF_S(v_S) \\
&= \int_0^1 \mathbf{y}(F_B(v_B), F_S(v_S))\, dq_S
\end{aligned}$$

Whereas,

$$\mathbf{p}(v_B;\, \tilde{\mathbf{x}}(\cdot,\cdot)) = \int_0^1 \mathbf{y}\left(\tilde{F}_B(v_B), F_S(v_S)\right) dq_S$$

Hence, $\mathbf{p}\left(v_B;\ \tilde{\mathbf{x}}(\cdot,\cdot)\right) = \mathbf{p}\left(\boldsymbol{v}_B\left(\tilde{F}(v_B)\right);\ \mathbf{x}^*(\cdot,\cdot)\right)$. We have that $\mathbf{p}\left(v_B;\ \mathbf{x}^*(\cdot,\cdot)\right)$ is an increasing function of v_B, and by FOSD, $\boldsymbol{v}_B\left(\tilde{F}(v_B)\right) \leqslant v_B$, hence

$$\mathbf{p}\left(v_B;\ \tilde{\mathbf{x}}(\cdot,\cdot)\right) \leqslant \mathbf{p}\left(v_B;\ \mathbf{x}^*(\cdot,\cdot)\right)\ \forall v_B$$

This is slightly hard to follow; essentially what we are saying is that, if we have fixed $\mathbf{x}^*(\cdot,\cdot), \tilde{\mathbf{x}}(\cdot,\cdot)$ to have the same quantile trade function $\mathbf{y}(\cdot,\cdot)$, then since quantiles of $\tilde{F}$ are higher than those of F, marginal trade probabilities of any fixed value v_B are lower for $\tilde{\mathbf{x}}(\cdot,\cdot), \tilde{F}$ than $\mathbf{x}^*(\cdot,\cdot), F$.

Construct the inverse functions as $\bar{v}_B\left(p;\ \mathbf{x}^*(\cdot,\cdot)\right) = \mathbf{p}^{-1}\left(v_B;\ \mathbf{x}^*(\cdot,\cdot)\right)$, and likewise for $\bar{v}_B\left(p;\ \tilde{\mathbf{x}}(\cdot,\cdot)\right)$. Then, we have that:

$$\bar{v}_B\left(p;\ \tilde{\mathbf{x}}(\cdot,\cdot)\right) \geqslant \bar{v}_B\left(p;\ \mathbf{x}^*(\cdot,\cdot)\right)\ \forall p \tag{4}$$

Finally, note that

$$\bar{v}_B\left(p;\ \mathbf{x}^*(\cdot,\cdot)\right) = \left\{v_B :\ \int_0^1 \mathbf{y}\left(F_B(v_B), F_S(v_S)\right) dq_S = p\right\}$$

$$\bar{v}_B\left(p;\ \tilde{\mathbf{x}}(\cdot,\cdot)\right) = \left\{v_B :\ \int_0^1 \mathbf{y}\left(\tilde{F}_B(v_B), F_S(v_S)\right) dq_S = p\right\}$$

This immediately implies that

$$F_B\left(\bar{v}_B\left(p;\ \mathbf{x}^*(\cdot,\cdot)\right)\right) = \tilde{F}_B\left(\bar{v}_B\left(p;\ \tilde{\mathbf{x}}(\cdot,\cdot)\right)\right)\ \forall p \tag{5}$$

Finally, we are done. Combining inequality (4) and equality (5), and plugging into expression (3) for $BRev$, we have that

$$BRev\left(\mathbf{x}^*(\cdot,\cdot), F_B(\cdot)\right) = \int_0^1 \bar{v}_B\left(p;\ \mathbf{x}^*(\cdot,\cdot)\right)\left(1 - F_B\left(\bar{v}_B(p)\right)\right)\, dp$$

$$\leqslant \int_0^1 \bar{v}_B\left(p;\ \tilde{\mathbf{x}}(\cdot,\cdot)\right)\left(1 - \tilde{F}_B\left(\bar{v}_B(p)\right)\right)\, dp = BRev\left(\tilde{\mathbf{x}}(\cdot,\cdot), \tilde{F}_B(\cdot)\right)$$

This proves the inequality in (2), proving part 2 of Claim 2 and thus the original Claim 1.

References

[1] Roger B Myerson and Mark A Satterthwaite. Efficient mechanisms for bilateral trading. *Journal of Economic Theory*, 29(2):265 – 281, 1983.