Online Appendix to "Competition and Manipulation in Derivative Contract Markets"

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This is the online appendix for "Competition and Manipulation in Derivative Contract Markets". Proofs for all sections are presented in section 6.

1 General functional forms

The expression for the pass-through of contract positions to demand in an agent's best-response bid curve, from (1) of proposition 1 in the main text, is:

$$\frac{\partial z_{Di}}{\partial y_{ci}} = \frac{\kappa}{\kappa + d}, \tag{1}$$

This expression approximately generalizes to arbitrary utility functions and random residual supply functions. Suppose that an agent with utility $u(z)$, holding y_c contracts, submits bid function $z(p)$ facing a general random nonlinear residual supply function. As in Wilson (1979), assume that, given the agent's bid function, I describe residual supply by a price distribution function $H(p; z(p))$, where $H(p; z(p))$ is a CDF over prices for any bid function $z(p)$. For any given realization of price, the agent's utility is

$$U(z, p) = u(z) - pz(p) + py_c$$

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Hence, the agent's expected utility given bid function $z(p)$ is:

$$\int u(z(p)) - pz(p) + py_c \, dH(p, z(p)) \tag{2}$$

and the agent chooses $z(p)$ to maximize (2).

Proposition 1. *A necessary first-order condition for optimal bid functions* $z_D(p)$ *is:*

$$\left[u'(z_D(p)) - p\right] + (y_c - z_D(p)) \, G(p, z_D(p)) = 0 \tag{3}$$

Where, $G(p, z_D(p))$ *is the ratio*

$$G(p, z_D(p)) \equiv -\frac{H_z(p, z_D(p))}{H_p(p, z_D(p))} \tag{4}$$

In words, $G(p, z_D(p))$ is the amount that p has to increase to ensure that the probability that the price falls below p is constant, as we increase z. Thus, it can be thought of as the average slope of residual inverse supply through the point $(p, z_D(p))$. When the slope of demand is constant at d, we have $G(p, z_D(p)) = \frac{1}{d}$. The equation (3) can then be interpreted as a markup formula, relating the difference between marginal utility and price, $u'(z_D(p)) - p$, to the difference between trade quantity and contracts, $y_c - z_D(p)$, multiplied by the slope of inverse residual supply $G(p, z_D(p))$.

Differentiating (3) and using the implicit function theorem, we have:

$$\frac{dz_D}{dy_c} = \frac{G(p, z_D(p))}{G(p, z_D(p)) - u''(z_D(p)) + G_z(p, z_D(p)) \, (z_D(p) - y_c)} \tag{5}$$

The term $G_z(p, z(p))$ represents the curvature of inverse residual supply – that is, as we increase $z(p)$, whether the slopes of inverse residual supply through the points $(p, z(p))$ increases or decreases. The sign of $G_z(p, z(p))$ is ambiguous. Intuitively, the pass-through $\frac{dz_D}{dy_c}$ is lower for net buyers, who have $z_D(p) - y_c > 0$, when G_z is positive, so inverse residual supply is more convex; symmetrically, pass-through is lower for net sellers, who have $z_D(p) - y_c < 0$, if inverse residual supply is more concave[1]. Expression

[1] This is related to the theory of pass-through under market power, discussed in Weyl and Fabinger (2013).

(5) simplifies if we disregard this curvature term $G_z(p, z_D(p))$:

$$\frac{dz_D}{dy_c} \approx \frac{G(p, z_D(p))}{G(p, z_D(p)) - u''(z_D(p))} \tag{6}$$

Specializing to the linear-quadratic case of the baseline model, plugging in $u''(z) = -\frac{1}{\kappa}$, $G(p, z(p)) = \frac{1}{d}$, expression (6) simplifies to $\frac{\kappa}{\kappa+d}$, which is exactly the pass-through $\frac{dz_D}{dy_c}$ in the baseline model. Hence, (6) shows that characterization of manipulation incentives in proposition 1 in the main text approximately generalizes to arbitrary smooth utility and residual supply functions.

1.1 Measurement

Suppose that the econometrician observes multiple auction instances, and wishes to estimate the pass-through of contracts into submitted bid curves, as in expression (5), or the simpler version (6). A large body of recent work, summarized in Hortaçsu (2011), discusses how the distribution of inverse residual supply, $H(p, z_D(p))$, can be estimated, even with a relatively small number of auction observations, by using resampling methods to increase the effective size of the dataset. The estimate of $H(p, z)$ can be differentiated to recover $H_z(p, z)$ and $H_p(p, z)$, and then used to derive an estimate of the function $G(p, z_D(p))$ using (4)[2]. In principle, even the higher-order derivatives $G_z(p, z)$ and $G_p(p, z)$ can be estimated, although these estimates are likely to be imprecise.

In addition to estimates of the functions $G(p, z_D(p)), G_z(p, z), G_p(p, z)$, in order to calculate (5) or (6), the econometrician needs an estimate of the slope of agents' marginal utility, $u''(z)$[3]. Intuitively, the inverse of the slope of agents' submitted bid curves should be close to the slope of marginal utility, differing to the extent that agents are shading bids. As discussed in Hortaçsu (2011), given $G(p, z_D(p))$, the agent's Euler-Lagrange first-order condition (3) for optimal bidding can be inverted to recover the marginal utility function $u'(z)$. The slope of marginal utility $u''(z)$ can be estimated similarly.

[2]From an econometric standpoint, it may be preferable to directly estimate the derivatives $H_z(p, z_D(p))$ and $H_p(p, z_D(p))$ rather than to estimate the function $H(p, z)$ and then take its derivatives. For example, Larsen and Zhang (2018) discusses how local polynomial regression can be used to nonparametrically estimate ratios of derivatives.

[3]Note that I call $u''(z)$ the "slope of marginal utility," which is equivalent to the "curvature of utility." The slope of marginal utility is more intuitive in my context, as it is the inverse of the slope of agents' truthful demands, which is κ in the baseline model.

Claim 1. The slope of agent's marginal utility, $u''(z_D(p))$, at any point $z_D(p)$, satisfies:

$$\frac{dz_D}{dp} = \frac{1 - (y_c - z_D(p))\, G_p(p, z_D(p))}{u''(z_D(p)) - G(p, z_D(p)) + (y_c - z_D(p))\, G_z(p, z_D(p))} \tag{7}$$

If the curvature terms $G_p(p, z_D(p))$ and $(y_c - z_D(p))\, G_z(p, z_D(p))$ are ignored, this approximately simplifies to:

$$u''(z_D(p)) \approx G(p, z_D(p)) - \frac{1}{z_D'(p)} \tag{8}$$

Plugging (7) or (8) and an estimate of $G(p, z)$ into (5) or (6) allows us to estimate the pass-through $\frac{dz_D}{dy_c}$ of contract positions into bids in general nonlinear settings.

2 Endogeneous contract positions and welfare

In this section, I endogenize agents' contract positions, by assuming that agents contract based on an auction to imperfectly hedge against random exposures to an uncertain state of the world. The model shows that manipulation decreases total welfare through two channels: by decreasing the effectiveness as the auction price as a hedge, increasing agents' exposure to fundamental risk, and by decreasing allocative efficiency in the auction.

2.1 Model

There is a countably infinite collection of agents, $i \in \{1, 2, \ldots \infty\}$. I assume that agents have identical demand slopes, so $\kappa_i = \kappa\ \forall i$. Each agent holds a productive asset which produces an uncertain quantity $x_i \pi$ of income, measured in consumption units. $\pi \sim N(\mu_\pi, \sigma_\pi^2)$ represents an uncertain state of the world. x_i represents agent i's exposure to π; I assume that $x_i \sim N(0, \sigma_x^2)$. Since the mean of x_i is 0, agents are equally likely to have incomes positively and negatively correlated with π; since there is an infinite number of agents, there is no aggregate risk in the economy. Agents have CARA utility over consumption, with risk aversion parameter α. That is,

$$u(c) = -\exp[-\alpha c]$$

This implies that an uncertain consumption bundle which is normally distributed with mean μ and variance σ^2 is utility-equivalent to a certain consumption bundle of size $\mu - \frac{\alpha\sigma^2}{2}$.

We assume that agents can only hedge against risk by holding contracts. This is because, as I describe in subsection 3.1 in the main text, I assume that agents observe their own utility functions, but are not able to distinguish between demand shocks y_{di} and the state of the world π. Thus, agents cannot contract directly on π; instead, they use an auction for some underlying good z, whose value is close to π, to attempt to elicit π from auction participants in an incentive-compatible manner. Specifically, as in the baseline model, the consumption value of z to agents is

$$\pi z + \frac{y_{di} z}{\kappa} - \frac{z^2}{2\kappa}$$

where $y_{di} \sim N\left(0, \sigma_d^2\right)$ is a demand shock. As discussed in subsection 3.1 in the main text, I assume that the term $\frac{z^2}{2\kappa}$ represents physical holding costs or decreasing returns for the underlying asset, rather than agents' reluctance to take large positions in the asset because of risk aversion.

Agents play a multi stage game, which I will refer to as the *hedging auction game*. The game proceeds as follows:

1. **Exposures,** x_i**:** Agents' exposures $x_i \sim N\left(0, \sigma_x^2\right)$ are realized and privately observed.

2. **Contract purchasing,** y_{ci}**:** Agents decide how many unit of contracts, y_{ci}, to purchase at price $E(\pi)$ from a market maker.

3. **State of nature,** π**:** The state of nature $\pi \sim N\left(0, \sigma_\pi^2\right)$ is realized.

4. **Benchmark determination:** n agents are randomly selected to participate in an auction to determine the benchmark; they draw demand shocks y_{di}, and bid to determine their allocations z_i, auction payments $z_i p_b$, and the price p_b.

5. **Contract settlement:** All agents receive their asset payments $x_i \pi$ and their contract payments $y_{ci} p_b$, and the game ends.

As in the baseline model, I assume that agents do not separately observe π and y_{di}. This implies that agents are unable to contract on π directly; π must be elicited from agents by running an auction. The auction price is a noisy signal of π both because of agents' demand shocks y_{di} and because of agents' manipulation incentives.

I assume that not all hedging agents participate in the auction. In practice, a large number of agents will use contract markets to hedge against aggregate uncertainty. However, the good whose price is used to set the benchmark is often very specific – for example, wheat futures contracts used internationally to hedge against aggregate price fluctuations is based on delivery of wheat of particular grades, for delivery in Chicago. Similarly, interest rate benchmarks such as SOFR, designed for use in a large variety of floating-rate debt, such as mortgages or student loans, are set based on yields of repo loans, a very particular kind of debt product. Hence, the assumption that a limited subset of hedging agents are active in the market for the underlying asset is empirically justified.

The assumption that n agents are randomly selected to participate in the auction is stylized; in practice, the set of agents who are willing to trade Chicago wheat, or repo loans, is fixed. An alternative model would fix a set of n agents who participate in auction. In such a model, auction participants purchase more contracts ex ante than nonparticipants, since they anticipate being able to move prices in their favor on average. Since this is a pure transfer, it does not affect aggregate efficiency; I assume this away to simplify the model.

Once the game concludes, given z, x_i, y_{ci}, p_b, agent i's utility is:

$$U(z, x_i, y_{ci}, p_b) = -\exp\left[-\alpha\left[\pi z + \frac{y_{di} z}{\kappa} - \frac{z^2}{2\kappa} - p_b z + \pi x_i + p_b y_{ci} - \mu_\pi y_{ci}\right]\right] \quad (9)$$

In words, the auction-based hedging games combines the auction in the baseline model to an endogeneous contract purchasing stage, in which agents purchase contracts in order to partially hedge idiosyncratic risk. I solve the game by analyzing agents' contract purchasing decisions in subsection 2.2, and then their auction bidding decisions in subsection 2.3, and characterizing an equilibrium between decisions in these two stages. I then analyze the welfare properties of equilibrium in subsection 2.4.

2.2 Contract purchasing

From the perspective of any individual agent, the probability of being chosen to participate in the auction game is 0, and the expected utility from participating in the auction game is finite. Hence, agents choose contract positions in stage 2 as if they will not participate in the auction; setting $z = 0$ in 9, in stage 2, agents choose contract positions y_{ci} to maximize the utility function:

$$U(z_i, x_i, y_{ci}, p_b) = -\exp\left[-\alpha\left[\pi x_i + p_b y_{ci} - \mu_\pi y_{ci}\right]\right]$$

Suppose that the agent forsees that, in stage 4, $Var(p_b) = \sigma_p^2$. In subsection 6.2.1, I show that an agent with exposure x_i to the state of the world optimally purchases:

$$y_{ci} = -x_i \frac{\sigma_\pi^2}{\sigma_\pi^2 + \sigma_p^2} \tag{10}$$

Since the ratio $\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_p^2} < 1$, the agent imperfectly hedges. $\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_p^2}$ increases towards 1, and thus y_{ci} gets closer to $-x_i$, as σ_p^2 decreases, so that the price is a better signal for π. Note also that, since contract purchase decisions are linear in exposure shocks x_i, contract positions sum to 0 across all agents in any equilibrium, and the market maker ends up with no net contract position and no net monetary transfer in any equilibrium; the stage 2 market maker always breaks even, so the game is well-defined.

Taking the variance of both sides of (10), we find the induced variance of contract positions y_{ci} across agents:

$$\sigma_c^2 = \sigma_x^2 \left(\frac{\sigma_\pi^2}{\sigma_\pi^2 + \sigma_p^2}\right)^2 \tag{11}$$

2.3 Benchmark determination

Once π is realized and observed by all agents, consumption and the asset z are both riskless; thus agents' utility in the stage 4 auction is quasilinear in z and income, and is exactly their utility in the baseline model of section 3 in the main text:

$$-y_{ci}\pi + \frac{y_{di}z}{\kappa} - \frac{z^2}{2\kappa} + \pi z - p_b z + p_b y_{ci}$$

Hence, agents bid as in the symmetric equilibrium described in Appendix A.7 in the main text. Agents' equilibrium demand functions are:

$$z_{Di}(p; y_{di}, y_{ci}) = \frac{n-2}{n-1} y_{di} + \frac{1}{n-1} y_{ci} - \frac{n-2}{n-1} \kappa (p - \pi)$$

Given y_{ci} for all agents, the market clearing price is:

$$p_b - \pi = \frac{1}{n\kappa}\sum_{i=1}^{n} y_{di} + \frac{1}{n(n-2)\kappa}\sum_{i=1}^{n} y_{ci} \tag{12}$$

Taking the variance of (12), we have:

$$\sigma_p^2 \equiv Var(p_b - \pi) = \frac{\sigma_d^2}{n\kappa^2} + \frac{\sigma_c^2}{n(n-2)^2\kappa^2} \tag{13}$$

Equations (11) and (13) are two relationships between σ_c^2 and σ_p^2, which must both be satisfied in equilibrium. In (13) σ_p^2 is increasing in σ_c^2. In (11), σ_c^2 is strictly decreasing in σ_p^2, with $\sigma_c^2 = \sigma_x^2$ when $\sigma_p^2 = 0$, and $\sigma_c^2 \to 0$ as $\sigma_p^2 \to \infty$. Hence there is a unique pair of positive σ_c^2, σ_p^2 values which satisfy (11) and (13), which characterize the unique equilibrium of the hedging auction game.

Proposition 2. *There is a unique equilibrium in the hedging auction game for any collection of parameters $n, \kappa, \sigma_x^2, \sigma_\pi^2$, in which σ_c^2, σ_p^2 satisfy (11) and (13):*

$$\sigma_c^2 = \sigma_x^2\left(\frac{\sigma_\pi^2}{\sigma_\pi^2 + \sigma_p^2}\right)^2$$

$$\sigma_p^2 = \frac{\sigma_d^2}{n\kappa^2} + \frac{\sigma_c^2}{n(n-2)^2\kappa^2}$$

Equations (11) and (13) describe the comparative statics of equilibrium behavior. (11) states that the extent to which agents hedge depends on $\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_p^2}$, which compares the variance of the state of nature, σ_π^2, to the variance in auction prices, σ_p^2. Intuitively, if the variance of auction prices is high, contracts are not an effective hedge against fundamental uncertainty, and agents buy less contracts per unit exposure shock x_i they receive. Equation (13), which follows directly from (98) in Appendix A.7 in the main text, states that price variance is higher when there are more contracts, less participants, and lower slopes of demand κ. Increasing n or κ increases equilibrium contract position size σ_c^2 by decreasing manipulation and thus improving the effectiveness of auction prices for hedging. Increasing σ_π^2 or σ_x^2 also increases equilibrium contract position size σ_c^2, by increasing the demand for hedging.

Henceforth, we will hold primitives $n, \kappa, \sigma_x^2, \sigma_\pi^2$ fixed, and use σ_p^2 and σ_c^2 to refer to the unique equilibrium values specified in proposition 2.

2.4 Hedging effectiveness

In the setting of the auction-based hedging game, there is no aggregate risk; if agents were allowed to write contracts that paid exactly π, all agents could perfectly hedge and achieve riskless portfolios with average payoff π. Moreover, since agents' utilities for the underlying asset are identical, regardless of the realization of π and the exposure shocks x_i, the optimal allocation involves no trade. The equilibrium of the auction-based hedging game deviates from the optimal allocation in both respects. First, manipulation makes the auction price a noisier signal of the state of nature π, so all agents are unable to fully hedge and lose utility from increased portfolio risk. Second, the n agents who participate in the auction accumulate nonzero net positions in the asset in order to move prices favorably, causing a reduction in allocative efficiency.

If an agent has contract position x_i, her portfolio has variance

$$x_i^2 \frac{\sigma_p^2 \sigma_\pi^2}{\sigma_\pi^2 + \sigma_p^2}$$

where σ_p^2 is the equilibrium variance of auction prices. Since agents have CARA utility with risk aversion parameter α, this causes a loss in ex ante expected utility. Taking expectations over x_i, the expected loss in expected utility, measured in consumption units, is:

$$\frac{\alpha \sigma_x^2}{2} \left(\frac{\sigma_p^2 \sigma_\pi^2}{\sigma_\pi^2 + \sigma_p^2} \right)^2$$

Even in the absence of manipulation, the auction price is a noisy signal for π, because there is a finite number of auction participants who are subject to demand shocks. For comparison, assume that all auction participants bid truthfully, submitting bid curves equal to their marginal utility for the asset, which are affected by demand shocks y_{di} but not by contract positions y_{ci}. In subsection 6.2.2, I show that the variance of benchmark prices under truthful bidding is:

$$\sigma_p^2 = \frac{\sigma_d^2}{n\kappa^2}$$

In principle, the variance of benchmark prices in equilibrium differs from the variance under truthful bidding for two reasons: agents' responses to demand shocks y_{di} differ, and agents' contract positions y_{ci} affect bids. Interestingly, if agents hold no contract positions and bids are only affected by demand shocks, the variance of prices in equilibrium is exactly the same as in the truthful-bidding case. This is because there are two effects of market power on agents' response to demand shocks: agents' demand shocks pass through to bids less, decreasing the variance of prices, and agents bid less elastic bid curves, increasing the variance of prices. In equilibrium, these two effects exactly offset each other, and the variance of prices is unchanged from the truthful-bidding case. While this finding may not be robust to more general models, it suggests at least that the effect of market power on price dispersion, without considering manipulation, is ambiguous.

In contrast, contract positions and manipulation unambiguously increase price dispersion. Subsection 6.2.2 shows that the variance of prices under both demand shocks and contract positions is:

$$\sigma_p^2 = \frac{\sigma_d^2}{n\kappa^2} + \frac{\sigma_c^2}{n\left(n-2\right)^2\kappa^2}$$

Thus, manipulation adds variance to auction prices around π, decreasing the effectiveness of contracts for hedging against uncertainty in π and thus decreasing aggregate welfare.

2.5 Allocative efficiency

The second source of welfare loss is that manipulation distorts the allocations of the n agents participating in the auction. Due to demand shocks, the n agents enter into the auction with different utilities for the asset, and gains from trade are possible; due to the presence of asymmetric information, fully efficient reallocation is impossible, and some welfare losses are unavoidable. For comparison, I first assume that agents in the auction receive demand shocks but hold no contract positions. In subsection 6.2.3, I show that agents' expected gains from trade are, under fully truthful bidding,

$$\frac{\sigma_d^2}{\kappa}\left(\frac{n-1}{n} - \frac{1}{2\left(n-1\right)}\right) \tag{14}$$

In equilibrium, expected welfare is

$$\frac{\sigma_d^2}{\kappa}\left(\frac{n-2}{n} - \frac{\left(n-2\right)^2}{2n^2\left(n-1\right)}\right) \tag{15}$$

The difference between these two is $\frac{n-2}{n^2}$, which is strictly positive for $n \geqslant 3$. Thus, market power lowers expected welfare, by inhibiting efficient reallocation of the asset.

From subsection 6.2.3, expected allocative welfare when agents hold contract positions and manipulate is:

$$\frac{\sigma_d^2}{\kappa}\left(\frac{n-2}{n}-\frac{(n-2)^2}{2n^2(n-1)}\right)-\frac{\sigma_c^2}{\kappa}\left(\frac{(n-2)^2+1}{2n^2(n-2)^2}\right) \tag{16}$$

There is an additional distortion term in (16), relative to (15), which further lowers agents' welfare. Intuitively, agents' bids are affected by their contract positions, since they want to influence prices to increase their contract payoffs. Since contract positions sum to 0 across the entire set of agents, the profits of manipulators are pure transfers from other agents, and are not relevant for welfare. Thus, from a welfare standpoint, manipulation serves to add further independent noise to agents' allocations, about their distorted levels under market power, which further decreases allocative efficiency.

2.6 Discussion

The model of this section provides a simple framework for analyzing the welfare implications of price benchmarks. Agents who wish to hedge against an imperfectly observed state of the world do so by contracting on an auction of a good whose mean value to participants is π. The auction is a noisy signal of π both because of auction participants' demand shocks, and because auction participants have incentives to trade in the auction to push prices in their favor. Manipulation decreases aggregate welfare both by adding variance to prices, decreasing the effectiveness of the auction price for hedging against changes in π, and by decreasing allocative efficiency in the auction.

The fact that there are two distinct sources of welfare losses from manipulation complicates welfare analysis of market outcomes. Qualitatively, more manipulation tends to decrease both kinds of welfare, but the features of manipulation that matter in each case are subtlely different. The variance of benchmark prices is all that matters for hedging effectiveness in the contract market;[4] benchmark bias does not influence hedging effectiveness. For example, if underlying market participants systematically hold short contract positions, manipulation will lower the benchmark in expectation; however,

[4]This depends on the assumption that all variables are normally distributed – in general, other features of the distribution may also matter.

if this is foreseen *ex ante*, the *ex ante* price of the contract will adjust to account for expected bias, and agents' ability to hedge uncertainty in π using contracts is unaffected. However, in this example, manipulation still creates allocative distortions: during contract settlement, in order to move settlement prices downwards, market participants must sell the underlying asset beyond their fundamental demand to do so, distorting allocations of the underlying asset. Similarly, if there are multiple manipulators who make exactly offsetting manipulative trades every time contracts settle, contract hedging effectiveness is unaffected, but allocative efficiency in the underlying market decreases.

It is thus difficult to construct a welfare metric which simultaneously accounts for welfare from hedging in the contract market and allocative efficiency in the underlying market. In the main text of the paper, I primarily focus on manipulation-induced benchmark variance as a proxy metric for welfare. This can be thought of as putting full weight on welfare from hedging effectiveness and no weight on allocative efficiency. This may not be appropriate for settings in which the distortions that manipulation creates in underlying markets are important.

3 Interdependent values

In the main text of the paper, I assumed agents have fully private values. In this subsection, I allow agents to have partially interdependent values. There are a number of papers studying on linear multi-unit double auctions with interdependent values (Du and Zhu, 2017; Rostek and Weretka, 2012; Vives, 2011). In this section, I study a relatively simple model of interdependent values, loosely based on the model of Vives (2011).

3.1 Model

There are n symmetric agents, with utility functions:

$$U(z,p) = \pi z + \frac{y_{di}}{\kappa} z + \frac{\theta \sum_{j \neq i} y_{dj}}{n\kappa} z - \frac{z^2}{2\kappa} + p y_{ci} - zp \tag{17}$$

Assume that $\theta \in (0,1)$, and assume that agents have identically distributed normal demand shocks and contract positions, $y_{di} \sim N\left(0, \sigma_d^2\right), y_{ci} \sim N\left(0, \sigma_c^2\right)$. The difference from the baseline model is the term $\frac{\theta \sum_{j \neq i} y_{dj}}{\kappa}$: agents' utilities depend not only on their own demand shocks y_{di}, but also on other agents' demand shocks y_{jd}, proportional to a factor θ, with $0 < \theta < 1$. Subsection 6.3.1 shows how this utility function can be derived from a partially common-valued model in which agents imperfectly observe their own value for the asset, as in Vives (2011).

3.2 Equilibrium

We conjecture the existence of a linear equilibrium, in which residual supply facing agent i is:

$$z_{RS}(p) = (p - \pi)\, d + \eta \tag{18}$$

When values are interdependent, agent i cares about the level of residual supply. This is because higher levels of η imply that the demand functions $z_{Dj}(p)$ submitted by other agents are lower, so other agents are trying to sell; this implies that other agents' demand shocks y_{dj} are likely to be low, which decreases agent i's value for the asset. Agent i's optimal bidding strategy depends on $E\left[\frac{\theta}{n}\sum_{j\neq i} y_{jd} \mid \eta\right]$; in equilibrium, this will be linear in η, so we can define:

$$\alpha \equiv \frac{E\left[\frac{\theta}{n}\sum_{j\neq i} y_{jd} \mid \eta\right]}{\eta} \tag{19}$$

When α is higher, adverse selection is worse, because shifts in residual supply are more correlated with the expected value of other agents' demand shocks. Fixing α, agents' expected utility for purchasing z when the auction price is p is:

$$E\left[U(z,p) \mid \eta\right] = \pi z + \frac{y_{di}}{\kappa} z + \frac{E\left[\frac{\theta}{n}\sum_{j\neq i} y_{dj} \mid \eta\right]}{\kappa} z - \frac{z^2}{2\kappa} + p y_c - zp$$

$$E\left[U(z,p) \mid \eta\right] = \pi z + \frac{y_{di}}{\kappa} z + \frac{\alpha\eta}{\kappa} z - \frac{z^2}{2\kappa} + p y_c - zp \tag{20}$$

Agents' submitted bid curves must maximize (20). In contrast to the baseline model, agents' bid curves will no longer be ex-post best responses with respect to other agents' y_{di}, y_{ci} values; this is because agents cannot distinguish between shocks to η that result from demand shocks or contracts. The best that agents can do is to submit bid curves which achieve the optimal choice of $(p, z_{RSi}(p,\eta))$ every possible realization of η. In other words, the bid curve $z_{Di}(p; y_{di}, y_{ci})$ must pass through all points $(p^*(\eta), z_{RSi}(p^*(\eta), \eta))$ which satisfy:

$$p^*(\eta) = \arg\max_p E\left[U(z_{RSi}(p), p) \mid \eta\right] \tag{21}$$

Hence, equilibrium bid curves must simultaneously satisfy (19) and (21). Subsection 6.3.2 shows that these two conditions are equivalent to a system of equations described in the following proposition.

Proposition 3. *Equilibrium values of* α *and* d *under interdependent values must satisfy the following two equations:*

$$\text{Best Response}: \ d = \frac{(n-2)\,\kappa}{1+n\alpha} \tag{22}$$

$$\text{Endogeneous Toxicity}: \ \alpha = \frac{d^2\sigma_d^2}{\kappa^2\sigma_c^2 + d^2\sigma_d^2}\frac{\theta}{n} \tag{23}$$

Equilibrium bid curves are:

$$z_D\,(p) = \frac{dy_d + \kappa y_c - d\,(\kappa - \alpha d)\,(p-\pi)}{\alpha d + \kappa + d} \tag{24}$$

3.3 Comparative statics

Figure 1 depicts the best response and endogeneous toxicity equations characterizing equilibrium. The best response equation, (22), describes the slope of residual supply induced by agents' best response bidding behavior, fixing a given toxicity of order flow α. If α is higher, adverse selection is worse, meaning that upwards shifts in residual supply η facing a given agent i are more correlated with other agents' negative demand shocks; agent i thus bids less aggressively, so the slope of residual supply d must be lower. Thus, the best response equation (22) specifies a decreasing relationship between d and α; in particular, $d = (n-2)\,\kappa$ when $\alpha = 0$ and $d \to 0$ as $\alpha \to \infty$. As a result, the equilibrium value of d will always be lower than $(n-2)\,\kappa$, the slope of residual supply without adverse selection – this reproduces the intuition from Du and Zhu (2012) that adverse selection causes agents to bid less aggressively.

The endogeneous toxicity equation, (23), describes average toxicity for a given level of d. When the slope of residual supply is low, agents' manipulation incentives are high, so shifts in residual supply η reflect contract positions more and demand shocks less; this implies that shifts in residual supply are less toxic, or equivalently that α is lower. Thus, the endogeneous toxicity equation (23) specifies an increasing relationship between d and α; in particular, $d = 0$ when $\alpha = 0$ and $d \to \infty$ as $\alpha \to \theta$. Thus, for any collection of primitives, there is a unique pair of points d, α which satisfy (22) and (23).

The primary effect of interdependent values is that it lowers the level of d, the equilibrium slope of residual supply for any given level of κ. In equilibrium, $d = \frac{(n-2)\kappa}{1+n\alpha}$, which is lower than the slope $d = (n-2)\,\kappa$ when values are fully private. From (24), the relative impact of demand shocks and contracts on bidding is still governed by the ratio of d to κ; since d is lower under interdependent values, agents manipulate more per unit contract that they hold under interdependent values than under private values.

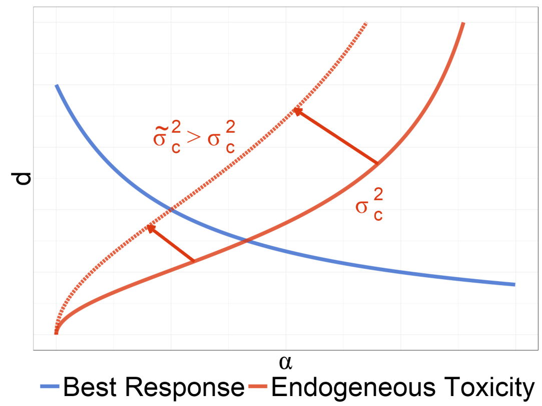
If we increase σ_c^2, then the locus of points satisfying the endogeneous toxicity equation (23) shifts upwards in $\alpha - d$ space, as shown in figure 1; this moves along the best response curve (22), increasing d and decreasing α in equilibrium. Intuitively, as contract volume increases, agents manipulate more; this means that most shocks to residual supply are due to manipulation rather than demand shocks. This makes adverse selection less of a problem, so agents bid more aggressively in equilibrium; this increases the slope of residual supply d and decreases the amount that agents manipulate per unit contract that they hold.

3.4 Discussion

The primary finding of this section is that adverse selection causes bid shading, which decreases the slope of residual supply; this increases the ratio $\frac{\kappa}{\kappa+d}$ and thus increases manipulation incentives. However, as manipulation increases, adverse selection decreases, in the sense that variation in the level of residual supply is more likely to be caused by manipulators, who have no information about the common value of the asset, rather than demand shocks. The intuition that manipulation decreases adverse selection was noted by Kumar and Seppi (1992), showing that manipulation decreases bid-ask spreads in a Kyle (1985) model of informed and uninformed trading with a competitive market maker.

A major idea in the main text of the paper is that equilibrium manipulation incentives are related to the ratio $\frac{\kappa}{\kappa+d}$, which can be estimated in data. The results of this section show that, in contexts where adverse selection is an important concern, the formula $\frac{\kappa}{\kappa+d}$ may understate manipulability; due to bid shading, the slope of agents' bid curves may be very different from their slopes of demand. Another way to put this is that manipulation is easier in settings with significant adverse selection, as a manipulator purchasing relatively small amounts can drive prices upwards significantly if other agents believe the manipulator has received a large positive demand shock, indicating that the common value of the asset is high. My framework is thus best-suited for classes of relatively homogeneous assets, for which information is close to symmetric across market

Figure 1: Best response and endogeneous toxicity curves



Notes. "Best Response" is the set of d, α points satisfying (22). "Endogeneous Toxicity" is the is the set of d, α satisfying (23). The dotted line is the endogeneous toxicity curve when σ_c^2 is increased to some value $\tilde{\sigma}_c^2 > \sigma_c^2$.

participants, hence trading is dominated by inventory concerns. For example, treasury bonds and other interest-rate products likely fit this classification. The magnitude of manipulation may be easier for classes of assets such as equities, in which asymmetric information across agents about fundamental values is an important concern.

4 Entry

In this section, I construct a simple model of manipulation with endogeneous auction paricipation. First, I characterize agents' profits from participating in auctions with n players. I assume that there are no demand shocks y_{di}, so all auction trade is induced by contract positions; this simplifies the analysis but does not qualitatively affect the results. I work with the symmetric model, in which all agents have identical slopes of demand κ.

As in the standard entry game model, I assume that identical agents $i = \{1 \ldots \infty\}$ sequentially decide whether to pay cost c to enter the auction, with all entry decisions commonly observed. Once an agent enters, $y_{ci} \sim N\left(0, \sigma_c^2\right)$ is realized, and the agent purchases y_{ci} contracts at the actuarially fair price of π. I thus take agents' contract positions as exogeneous; this is helpful for solving the model, because the marginal profit per unit contract is actually increasing, so agents are willing to purchase unbounded quantities of contracts at fixed prices.

I assume that y_{ci} is unobserved by all other agents. Once the $n+1$th agent decides not to enter, all agents who have entered, $i = \{1 \dots n\}$, play the symmetric auction equilibrium with n agents described in Appendix A.7 in the main text. In subsection 6.4, I show that, in an equilibrium with n agents, before contract positions y_{ci} are known, the expected utility of each agent is:

$$\frac{\sigma_c^2}{2\kappa (n-1)(n-2)} \tag{25}$$

This is strictly decreasing in n. If there is a fixed cost c for agents to draw $y_{ci} \sim N\left(0, \sigma_c^2\right)$ and participating in the benchmark setting auction, equilibrium in the entry game requires that agent sequentially enter until the expected profits of the marginal entrant are lower than cost. Letting n_{eq} represent the equilibrium number of entrants, the n_{eq}th entrant must makes positive profits and the $(n_{eq}+1)$th must make negative profits:

$$\frac{\sigma_c^2}{2\kappa (n_{eq}-1)(n_{eq}-2)} \geqslant c \tag{26}$$

$$\frac{\sigma_c^2}{2\kappa n_{eq} (n_{eq}-1)} \leqslant c \tag{27}$$

To simplify, if we assume (27) holds with equality, we have:

$$n_{eq} (n_{eq}-1) \approx \frac{\sigma_c^2}{2\kappa c} \tag{28}$$

(28) is intuitive: more agents enter when the cost of entry c is lower; when κ is lower, so agents can more easily trade to cause price impact and profit from contract manipulation; and when σ_c^2 is higher, so agents can more easily build up large contract positions.

Increasing the size of contract positions has ambiguous effects on equilibrium manipulation and the variance of benchmark prices. Holding fixed the number of entrants, n, increasing the expected size of each entrant's net contract position increases manipulation. However, increasing contract position size encourages more agents to enter attempting to manipulate; this increases the slope of residual supply in the auction, potentially decreasing the variance of prices in equilibrium. In the simple symmetric model I study here, the entry effect dominates, and increasing the size of contract positions in fact decreases equilibrium benchmark price variance. Since we assumed $Var(y_{ci}) = \sigma_c^2, Var(y_{di}) = 0$, the variance of prices in equilibrium with n_{eq} agents is

$$Var\left(p_b\right) = \frac{\sigma_c^2}{n_{eq}\left(n_{eq}-2\right)^2 \kappa^2} \tag{29}$$

To leading order, we have:

$$n_{eq}\left(n_{eq}-2\right)^2 \approx n_{eq}^3 \approx \left[n_{eq}\left(n_{eq}-1\right)\right]^{\frac{3}{2}}$$

Thus, we can substitute (28) into (29), to find the equilibrium variance of prices as a function of primitives σ_c^2, κ, c:

$$\left(\frac{\sigma_c^2}{\left(n-2\right)^2 n\kappa^2}\right) \approx \frac{\sigma_c^2}{\kappa^2\left(\frac{\sigma_c^2}{2\kappa c}\right)^{\frac{3}{2}}} = \frac{2\sqrt{2}c^{\frac{3}{2}}}{\sqrt{\kappa\sigma_c^2}}$$

The equilibrium variance of auction prices is thus asymptotically decreasing in the variance of agents' contract positions, σ_c^2; the effect of increased entry dominates the effect of increased manipulation. Figure 2 shows the behavior of the exact expressions for n and $Var\left(p_b\right)$ as σ_c^2 varies; for any interval on which n is fixed, increasing σ_c^2 increases $Var\left(p_b\right)$, but $Var\left(p_b\right)$ decreases whenever n increases.

4.1 Entry by non-manipulators

Manipulation also creates additional incentives for non-manipulators to participate in the auction. Consider an equilibrium with n agents with identical demand slopes κ. A single agent $i = 1$ is a non-manipulator; she holds no contract position, so $y_{ci} = 0$, and has a demand shock $y_{di} \sim N\left(0, \sigma_d^2\right)$. The other $n-1$ agents are pure manipulators, with no demand shocks, and contract positions with variance σ_c^2. In Appendix 6.4, I show that the utility of agent i, when she is facing an uncertain affine residual supply curve $z_{RSi}\left(p\right) = d\left(p-\pi\right)+\eta$, is:

$$\underbrace{\frac{\kappa^2\sigma_\eta^2}{2d\kappa\left(d+2\kappa\right)}}_{\text{RS Uncertainty}} + \underbrace{\frac{d^2\sigma_d^2}{2d\kappa\left(d+2\kappa\right)}}_{\text{Demand Shock}} \tag{30}$$

In words, agent i receives utility from two sources. The "RS Uncertainty" term represents agent i's utility from uncertainty in residual supply. Agent i prefers σ_η^2 to be higher, or residual supply to be more uncertain: intuitively, if residual supply is subject to random shocks, agent i benefits from being able to buy below her value and sell above her value. The "Demand Shock" term represents agent i's utility from trading to satisfy her demand shock. The coefficient

$$\frac{d^2}{2d\kappa\left(d+2\kappa\right)}$$

is an increasing function of d; intuitively, when agent i faces a more elastic residual supply curve, she has less price impact, and can more cheaply buy and sell to satisfy her demand shock, increasing her utility.

In equilibrium with $n-1$ manipulators, the slope of residual supply facing agent $i=1$ is $d=\left(n-2\right)\kappa$, and (54) in Appendix 6.4 shows that $\sigma_\eta^2=\frac{\sigma_c^2}{n-1}$. Plugging these expressions into (30), i's utility in equilibrium is:

$$\underbrace{\frac{\sigma_c^2}{2\left(n-2\right)\left(n-1\right)n\kappa}}_{\text{RS Uncertainty}}+\underbrace{\frac{n-2}{2n\kappa}\sigma_d^2}_{\text{Demand Shock}} \tag{31}$$

The effect of increasing n on utility, holding fixed all other terms, is ambiguous. Increasing n decreases the "RS Uncertainty" term, since it makes residual supply more certain as manipulators manipulate less, but increases the "demand shock" term as the slope of residual supply increases. Intuitively, if the demand shock σ_d^2 is small, agent i is primarily an arbitrageur, profiting when manipulation makes the price deviates from π; she prefers if the market is thin and prices often deviate from fundamentals. If the demand shock σ_d^2 is large, agent i is primarily a trader who wishes to liquidate her position; she prefers to be able to trade with minimal price impact.

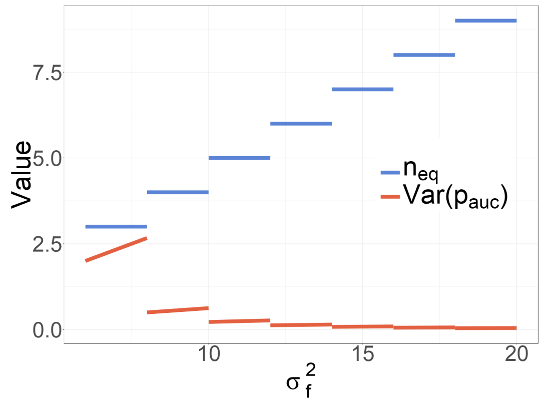
Increasing σ_c^2 holding n fixed unambiguously makes i better off. The "Demand Shock" term in utility is unaffected, and the "RS Uncertainty" term strictly increases. Intuitively, when agents other than i manipulate more, the location of residual supply is more uncertain, without affecting its slope. This means that non-manipulative traders can profit more from arbitraging shifts in the location of residual supply. While I do not solve a full equilibrium model of entry by manipulators and non-manipulators, intuitively, non-manipulators decrease the variance of prices by increasing the slope of residual supply, as well as trading against the average direction of manipulation; both effects counteract attempted manipulation, bringing auction prices closer to π.

4.2 Discussion

Designating an auction as the basis for contract settlement attracts entry into the auction. Manipulators enter in order to profit from manipulation, until the point at which auctions are sufficiently competitive that manipulation is unprofitable; non-manipulative traders are attracted to the auction, both by increased liquidity provided by manipulators, and by the possibility of profiting from arbitrage when manipulation causes prices to differ from fundamentals. This is consistent with empirical evidence showing that liquidity tends to increase significantly around benchmark-setting events. Stock trading is known to be concentrated around exchange opening and closing times; Admati and Pfleiderer (1988) develop a model in which liquidity traders' preference for thick markets endogeneously leads to clustering of trading over time. FX trading volume increases greatly at the WM/Reuters 4pm London fix; in most cases volume is 10 times greater than the daily mean (Financial Stability Board, 2014). In addition, Griffin and Shams (2018) show that volume in SPX options spikes when option prices are used to determine the VIX. This is consistent with my theory, in which participation and trade volume is higher when contract positions are larger. My theory also suggests that these increases in volume do not necessarily imply that agents are successfully manipulating, as entry tends to dampen the effects of manipulation.

The analysis of this subsection is stylized, in order to make the qualitative point that entry by manipulators and arbitrageurs counteracts attempted manipulation. When the assumptions of the model are relaxed – for example, if entrants are not symmetric, with different entry costs and slopes of demand – it is possible that the entry decision of any particular agent can increase benchmark variance. The idea that manipulation is tempered by competitive entry is also not novel to the literature. Kumar and Seppi (1992) show that, as the number of manipulators increases, competition drives profits from manipulation to 0. Hanson and Oprea (2008) show that manipulators increase the returns to informed trading, increasing incentives for information acquisition, so they can actually improve price discovery in equilibrium. My model is a private-values double auction model, in contrast to the common-values competitive market maker model studied in these two papers; the fact that both classes of models are able to reach qualitatively similar conclusions suggest that these results are relatively robust to modelling assumptions.

Figure 2: n and $Var(p_b)$ as functions of σ_c^2



Notes. Behavior of n and $Var(p_b)$ as σ_c^2 changes, with $\kappa = 1, c = 1, \sigma_d^2 = 0$.

Similar ideas about manipulation and competitive entry are also present in the classic literature on futures markets. Hieronymus (1977, pg. 328) writes: "Manipulation is its own best cure. To manipulate a price is to put it where it doesn't belong. The over-priced inventory or the underpriced commitment invariably leads to losses. . . . In the actively speculated markets the forces of countervailing power effectively prevent manipulation. It is only in the thin markets that power plays cause minor distortions which are profitable. A speculator of moderate scale commented, "Show me a market that someone has distorted and I will show you a way to make money, both with him and against him." The greater the level of speculation, the less is the amount of hanky-panky. Let them trade out."

5 Collusion

In this section, I show that colluding agents bid proportionately more per unit contract they hold, since they internalize the effects that price movements have on the profits of other agents within the cartel. I build on the asymmetric model of section 3 in the main text. As in the baseline model of the paper, suppose that n agents have demand slopes $\kappa_1 \dots \kappa_n$. Assume that some subset of agents $i \in \{1 \dots k\}$ bid collusively, and that they are able to perfectly transfer money and inventory among themselves once the auction concludes. Thus, colluding agents bid to maximize the joint physical utility of the total quantity $\bar{z}$ that they trade, plus any monetary payments they make. We can think of agents as bidding a single bid curve – any combination of individual bid curves which adds up to the same overall bid curve has the same effect for all agents.

Proposition 4. *If agents with* $\kappa_1 \dots \kappa_k$ *bid collusively, when they have demand shocks* y_{di} *and contract positions* y_{ci}*, they bid as if they are a single agent with demand shock* $\bar{y}_d = \sum_{i=1}^{k} y_{di}$*, contract position* $\bar{y}_c = \sum_{i=1}^{k} y_{ci}$*, and slope of demand* $\bar{\kappa} = \sum_{i=1}^{k} \kappa_i$*.*

Proposition 4 shows that agents bid as if they constitute a single agent with demand slope $\bar{\kappa}$, contract position $\bar{y}_c$, and net demand shock $\bar{y}_d$. Collusion increases manipulation – using the approximation of proposition 3 in the main text, the cartel trades approximately

$$\frac{\left(\sum_{i=1}^{k} y_c\right)\left(\sum_{i=1}^{k} \kappa_i\right)}{\sum_{i=1}^{n} \kappa_i} \tag{32}$$

units of the asset if the cartel's net contract position is $\bar{y}_c$, hence the contribution of manipulation to price variance is approximately

$$\frac{\left(\sum_{i=1}^{k} \kappa_i\right)^2 \sum_{i=1}^{k} \sigma_{ci}^2}{\left(\sum_{i=1}^{n} \kappa_i\right)^4} \tag{33}$$

If individuals bid separately, each individual agent i would only trade

$$\frac{\kappa_i}{\sum_{i=1}^{n} \kappa_i} y_{ci}$$

if her contract position is y_{ci}. Across agents $i \in \{1 \dots k\}$, this leads to total manipulation-driven trade of:

$$\frac{\sum_{i=1}^{k} \kappa_i y_{ci}}{\sum_{i=1}^{n} \kappa_i} \tag{34}$$

Manipulation-driven price variance is:

$$\frac{\sum_{i=1}^{k} \kappa_i^2 \sigma_{ci}^2}{\left(\sum_{i=1}^{n} \kappa_i\right)^4} \tag{35}$$

(33) is always larger than (35), hence variance of trade induced by cartel bidding is always larger than variance of trade induced by individuals' bidding. There are two related intuitions for why collusion increases manipulation incentives. First, colluding agents internalize the effects of increasing their own bids on other agents' contract profits.

Second, the cartel as a whole has a higher slope of demand than any individual cartel member, hence it can absorb large quantities of the underlying asset to move prices, spreading the quantity optimally among cartel members, at lower cost than any member could achieve by acting alone.

Many instances of benchmark manipulation observed in practice have involved collusion and communication between parties. In the case of LIBOR, which is determined by banks' unincentized announcements, traders communicated to each other whether they wanted higher or lower LIBOR rates (Ridley and Freifeld, 2015). In the case of the WM/Reuters FX fixing, which is determined by actual trades, traders pooled orders and planned how to trade during the fix to move the fixing in the desired direction (Levine, 2014).

The predictions of my theory in the main text rely strongly on the assumption that agents are independently optimizing; if agents are colluding, manipulation could in principle be arbitrarily large. In antitrust settings, while many features of regulation are basically structural, collusion is regulated on a primarily behavioral basis – it is illegal *per se* and is prosecuted based on "smoking gun" evidence, such as details of executives' communications. Similarly, contract market regulation could combine structural and behavioral approaches: structural tools could be used to limit predicted manipulation incentives, based on formal models of imperfect but non-collusive competition, and behavioral sanctions could be applied to compel market participants to behave imperfectly competitively, as specified in our formal models. Formal models of manipulation are also useful in diagnosing the economic effects of collusion. My results describe how large manipulation incentives are for a cartel of a given size; if regulators detect a cartel, these results inform regulators as to how much higher manipulation incentives in the cartel are relative to imperfect but non-collusive competition.

Formally modelling collusion is a subtle problem, as model outcomes can be very sensitive to details of assumptions of agents' ability to coordinate behavior and share profits, and agents' potential profits from deviating from cartel behavior. My model abstracts away from these issues, assuming that the cartel is able to maximize joint surplus and can efficiently redistribute inventory. This is the best possible case for collusion, so my model likely produces a high estimate for the extent to which agents are able to collude; in practice, there are various frictions and commitment issues which make collusion more difficult than my model suggests. An interesting direction for future research would be to develop more realistic models of collusive manipulation, accounting for imperfect profit-sharing and agents' incentives to deviate from cartel-specified behavior.

6 Proofs

6.1 Proofs for section 1

6.1.1 Proof of proposition 1

If an agent has utility function $u(z)$ and y_c contracts, her expected utility for bidding $z(p)$ is:

$$\int u(z(p)) - pz(p) + py_c \, dH(p, z(p))$$

Integrating by parts, and ignoring the constant term, we have:

$$\int -\left[u'(z(p))\, z'(p) - z(p) - pz'(p) + y_c\right] H(p, z(p))\, dp \tag{36}$$

The agent chooses $z(p)$ to maximize this. This is a calculus of variations problem. Define the integrand as:

$$F\left(p, z(p), z'(p)\right) = -\left[u'(z(p))\, z'(p) - z(p) - pz'(p) + y_c\right] H(p, z(p))$$

The Euler-Lagrange necessary conditions for $z_D(p)$ to optimize (36) are:

$$\frac{d}{dp} F_{z'} = F_z \tag{37}$$

Now,

$$F_{z'} = -\left[u'(z(p)) - p\right] H(p, z(p))$$

$$\frac{d}{dp} F_{z'} = -\Big[\left[u''(z(p))\, z'(p) - 1\right] H(p, z(p)) + \left[u'(z(p)) - p\right] H_p(p, z(p)) + \left[u'(z(p)) - p\right] H_z(p, z(p))\, z'(p)\Big]$$

$$F_z = -\Big[\left[u''(z(p))\,z'(p) - 1\right] H(p, z(p)) +$$
$$\left[u'(z(p))\,z'(p) - z(p) - pz'(p) + y_c\right] H_z(p, z(p)) \Big]$$

Plugging into (37), we have

$$\begin{aligned} &\left[u''(z(p))\,z'(p) - 1\right] H(p, z(p)) + \\ &\qquad \left[u'(z(p)) - p\right] H_p(p, z(p)) + \left[u'(z(p)) - p\right] H_z(p, z(p))\,z'(p) = \\ &\left[u''(z(p))\,z'(p) - 1\right] H(p, z(p)) + \left[u'(z(p))\,z'(p) - z(p) - pz'(p) + y_c\right] H_z(p, z(p)) \end{aligned}$$

Cancelling terms and rearranging, we get:

$$\left[u'(z_D(p)) - p\right] H_p(p, z_D(p)) + (z_D(p) - y_c)\, H_z(p, z_D(p)) = 0$$

Dividing by $H_p(p, z_D(p))$, and defining $G(p, z_D(p)) \equiv -\frac{H_z(p, z_D(p))}{H_p(p, z_D(p))}$, we can write:

$$\left[u'(z_D(p)) - p\right] + (y_c - z_D(p))\, G(p, z_D(p)) = 0 \tag{38}$$

where, $G(p, z_D(p))$ is the ratio

$$\frac{dp}{dz} = -\frac{\frac{dH}{dz}}{\frac{dH}{dp}}$$

as desired.

6.1.2 Proof of claim 1

Taking derivatives of (38) with respect to p, we have:

$$\left[u'(z_D(p)) - p\right] + (y_c - z_D(p))\, G(p, z_D(p)) = 0$$

$$\frac{\partial}{\partial p} : -1 + (y_c - z_D(p))\, G_p(p, z_D(p))$$
$$\frac{\partial}{\partial z} : \; u''(z_D(p)) - G(p, z_D(p)) + (y_c - z_D(p))\, G_Z(p, z_D(p))$$

Hence,

$$\frac{dz_D}{dp} = \frac{1 - (y_c - z_D(p))\, G_p(p, z_D(p))}{u''(z_D(p)) - G(p, z_D(p)) + (y_c - z_D(p))\, G_z(p, z_D(p))}$$

As before, if we disregard the curvature terms $G_p(p, z_D(p))$ and $G_z(p, z_D(p))$, we have:

$$\frac{dz_D}{dp} \approx \frac{1}{u''(z_D(p)) - G(p, z_D(p))}$$

Solving, we have

$$u''(z_D(p)) \approx G(p, z_D(p)) + \frac{1}{z'(p)}$$

as desired.

6.2 Proofs for section 2

6.2.1 Optimal contract positions

Utility is:

$$U(z_i, x_i, y_{ci}, p_b) = -\exp\left[-\alpha\left[\pi x_i + p_b y_{ci} - \mu_\pi y_{ci}\right]\right]$$

Since contracts are sold at the actuarially fair price of μ_π, the agent's utility is maximized by minimizing the variance of consumption, $\pi x_i + p_b y_{ci} - \mu_\pi y_{ci}$. We can write:

$$\pi x_i + p_b y_{ci} - \mu_\pi y_{ci} = \pi(x_i + y_{ci}) + (p_b - \pi)\, y_{ci} - \mu_\pi y_{ci}$$

In equilibrium, π is independent from $(p_b - \pi)$. Hence, with $\sigma_p^2 \equiv Var(p_b - \pi)$, the variance of consumption is:

$$(x_i + y_{ci})^2\, \sigma_\pi^2 + y_{ci}^2 \sigma_p^2 \tag{39}$$

Taking derivatives with respect to y_{ci} and setting to 0, we have:

$$2(x_i + y_{ci})\, \sigma_\pi^2 + 2 y_{ci} \sigma_p^2 = 0$$

Hence,

$$y_{ci} = -x_i \frac{\sigma_\pi^2}{\sigma_\pi^2 + \sigma_p^2} \tag{40}$$

Plugging (40) into (39) and simplifying, we find that portfolio variance, given x_i, is:

$$x_i^2 \frac{\sigma_p^2 \sigma_\pi^2}{\sigma_\pi^2 + \sigma_p^2}$$

6.2.2 Price dispersion in equilibrium

In equilibrium, demand for agent i is:

$$z_{Di}(p; y_{di}, y_{ci}) = \frac{n-2}{n-1} y_{di} + \frac{1}{n-1} y_{ci} - \frac{n-2}{n-1} \kappa (p - \pi)$$

The auction price is:

$$p_b - \pi = \frac{1}{n\kappa} \sum_{i=1}^{n} y_{di} + \frac{1}{n(n-2)\kappa} \sum_{i=1}^{n} y_{ci} \tag{41}$$

Substituting equilibrium price into demand and solving, the allocation for i is:

$$z = y_{di} \left(\frac{n-2}{n} \right) + y_{ci} \left(\frac{1}{n} \right) - \frac{n-2}{n-1} \left(\frac{1}{n} \sum_{j \neq i} y_{dj} + \frac{1}{n(n-2)} \sum_{j \neq i} y_{cj} \right) \tag{42}$$

To solve for the socially efficient allocation, note that we need marginal values to be equal, and total allocations to sum to 0:

$$U'(z) = \pi + \frac{y_{di}}{\kappa} - \frac{z}{\kappa} = \lambda$$

$$\sum z = 0$$

This implies that the efficient allocation has:

$$z = y_{di} - \frac{\sum_{i=1}^{n} y_{di}}{n} \tag{43}$$

In the case where agents bid honestly, price is equal to marginal utility. We then have:

$$U'(z) = \pi + \frac{\sum y_{di}}{n\kappa^2}$$

Which implies that

$$\sigma_p^2 = \frac{\sigma_d^2}{n\kappa^2}$$

Now in the inefficient case, with only market power, taking the variance of (41) setting $y_{ci} = 0$:

$$\sigma_p^2 = \frac{\sigma_d^2}{n\kappa^2}$$

If agents' contract positions also affect bidding, taking the variance of (41) including all terms:

$$\sigma_p^2 = \frac{\sigma_d^2}{n\kappa^2} + \frac{\sigma_c^2}{n\left(n-2\right)^2\kappa^2}$$

6.2.3 Allocative efficiency in equilibrium

Agents' utility, ignoring transfers, is:

$$u\left(z\right) = \pi z + \frac{y_{di}z}{\kappa} - \frac{z^2}{2\kappa} \tag{44}$$

Welfare in autarky is calculated by setting $z = 0$, but $u\left(0\right) = 0$, so gains from trade are simply equal to the expected value of (44). Using (43) and (42), we can calculate the distribution of z conditional on y_{di}; in the efficient case, this is:

$$z \sim N\left(\frac{n-1}{n}y_{di}, \frac{\sigma_d^2}{n-1}\right)$$

In equilibrium with demand shocks but no contracts, it is:

$$z \sim N\left(\frac{n-2}{n}y_{di}, \left(\frac{n-2}{n}\right)^2\frac{\sigma_d^2}{n-1}\right)$$

In equilibrium with contracts:

$$z \sim N\left(\frac{n-2}{n}y_{di}, \left(\frac{n-2}{n}\right)^2\frac{\sigma_d^2}{n-1} + \sigma_c^2\left(\frac{1}{n^2} + \frac{1}{n^2\left(n-2\right)^2}\right)\right)$$

Plugging these into (44) and taking expectations over y_{di}, we get average welfare in the efficient case:

$$\frac{\sigma_d^2}{\kappa}\left(\frac{n-1}{n}-\frac{1}{2(n-1)}\right) \tag{45}$$

The equilibrium case with demand shocks and no contracts:

$$\frac{\sigma_d^2}{\kappa}\left(\frac{n-2}{n}-\frac{(n-2)^2}{2n^2(n-1)}\right) \tag{46}$$

Note that the difference between (45) and (46) is

$$\frac{1}{n}-\frac{1}{2(n-1)}\left(1-\left(\frac{n-2}{n}\right)^2\right)=\frac{n-2}{n^2}$$

This is positive for $n > 3$ and decreasing in n, so the equilibrium is less efficient than the social optimum, as expected. Adding the effect of contracts, equilibrium expected welfare is:

$$\frac{\sigma_d^2}{\kappa}\left(\frac{n-2}{n}-\frac{(n-2)^2}{2n^2(n-1)}\right)-\frac{\sigma_c^2}{\kappa}\left(\frac{(n-2)^2+1}{2n^2(n-2)^2}\right) \tag{47}$$

Which is lower again than (46).

6.3 Proofs for section 3

6.3.1 Microfoundation

We construct a microfoundation of the utility function suppose that agents' utility is as in (17) using a model in which agents imperfectly observe their own values. Suppose that agents' utility is:

$$U=\pi z+\frac{\phi_i z}{\kappa}-\frac{z^2}{2\kappa}+py_c-zp$$

Assume that

$$\phi_i=\xi+\zeta_i$$

That is, ϕ_i consists of a common component, $\xi \sim N\left(0,\sigma_\xi^2\right)$, and a private valued component, $\zeta_i \sim N\left(0,\sigma_\zeta^2\right)$. ξ and all ζ_i are independent. Assume also that agents do not perfectly observe their own valuations; instead, they observe signals s_i, where

$$s_i=\phi_i+\epsilon_i$$

and $\epsilon_i \sim N\left(0, \sigma_\epsilon^2\right)$, and each ϵ_i is independent of ξ, ζ_i and all other ϵ_i.

The agent attempts to predict ϕ_i based on her own signal s_i and the sum of all other agents' signals $\sum_{j \neq i} s_j$. Consider the covariance matrix between $\phi_i, s_i, \sum s_j$:

$$\operatorname{Cov}\left(\begin{array}{c} \phi_i \\ s_{di} \\ \sum s_{dj} \end{array}\right) = \left(\begin{array}{ccc} \sigma_\xi^2 + \sigma_\zeta^2 & \sigma_\xi^2 + \sigma_\zeta^2 & (n-1)\,\sigma_\xi^2 \\ \sigma_\xi^2 + \sigma_\zeta^2 & \sigma_\xi^2 + \sigma_\zeta^2 + \sigma_\epsilon^2 & (n-1)\,\sigma_\xi^2 \\ (n-1)\,\sigma_\xi^2 & (n-1)\,\sigma_\xi^2 & (n-1)\left(\sigma_\zeta^2 + \sigma_\epsilon^2\right) + (n-1)^2\,\sigma_\xi^2 \end{array}\right)$$

By the projection formula for multivariate Gaussian random variables,

$$E\left[\phi_i \mid \left(\begin{array}{c} s_{di} \\ \sum s_{dj} \end{array}\right)\right] =$$
$$\left(\begin{array}{cc} \left[\sigma_\xi^2 + \sigma_\zeta^2\right] & \left[(n-1)\,\sigma_\xi^2\right] \end{array}\right) \left(\begin{array}{cc} \sigma_\xi^2 + \sigma_\zeta^2 + \sigma_\epsilon^2 & \sigma_\xi^2 \\ \sigma_\xi^2 & (n-1)\,\sigma_\xi^2 + \sigma_\zeta^2 \end{array}\right)^{-1} \left(\begin{array}{c} y_{di} \\ \sum y_{dj} \end{array}\right)$$

Inverting the middle matrix and multiplying through, this is:

$$E\left[\phi_i \mid \left(\begin{array}{c} s_{di} \\ \sum s_{dj} \end{array}\right)\right] = \frac{\left[\sigma_\epsilon^2\left(\sigma_\zeta^2 + \sigma_\xi^2\right) + \sigma_\zeta^2\left(\sigma_\zeta^2 + n\sigma_\xi^2\right)\right] s_{di} + \left[\sigma_\epsilon^2 \sigma_\xi^2\right] \sum_{j \neq i} s_{dj}}{\left(\sigma_\epsilon^2 + \sigma_\zeta^2\right)\left(\sigma_\epsilon^2 + \sigma_\zeta^2 + n\sigma_\xi^2\right)}$$

The coefficient on s_{di} is weakly larger than the coefficient on s_{dj}; the coefficients are equal only in a pure common values model, where $\sigma_\zeta^2 = 0$. Now, define

$$y_{di} = \frac{\left[\sigma_\epsilon^2\left(\sigma_\zeta^2 + \sigma_\xi^2\right) + \sigma_\zeta^2\left(\sigma_\zeta^2 + n\sigma_\xi^2\right)\right]}{\left(\sigma_\epsilon^2 + \sigma_\zeta^2\right)\left(\sigma_\epsilon^2 + \sigma_\zeta^2 + n\sigma_\xi^2\right)} s_{di}$$

Then, we have

$$E\left[\phi_i \mid y_{di}, \sum_{j \neq i} y_{dj}\right] = y_{di} + \theta \frac{\sum_{j \neq i} y_{dj}}{n}$$

$$\theta = \frac{n\sigma_\epsilon^2 \sigma_\xi^2}{\left(\sigma_\epsilon^2 + \sigma_\zeta^2\right)\left(\sigma_\epsilon^2 + \sigma_\zeta^2 + n\sigma_\xi^2\right)} \in (0,1)$$

Hence,

$$\begin{aligned}
E\left[U \mid y_{di}, \sum_{j\neq i} y_{dj}\right] &= \pi z + \frac{z}{\kappa} E\left[\phi_i \mid y_{di}, \sum_{j\neq i} y_{dj}\right] - \frac{z^2}{2\kappa} + p y_c - z p \\
&= \pi z + \frac{y_{di}}{\kappa} z + \frac{\theta \sum_{j\neq i} y_{dj}}{n\kappa} z - \frac{z^2}{2\kappa} + p y_c - z p
\end{aligned}$$

as desired.

6.3.2 Proof of proposition 3

From the perspective of agent i, residual supply in equilibrium is:

$$z_{RS}(p) = (p - \pi) d + \eta \tag{48}$$

Fix α, so that $\alpha \equiv E\left[\frac{\theta}{n} \sum_{j\neq i} y_{jd} \mid \eta\right]$. To satisfy (21), agent maximizes, pointwise in η,:

$$U = \pi z + \frac{\alpha\eta}{\kappa} z + \frac{y_{di} z}{\kappa} - \frac{z^2}{2\kappa} + p y_c - z p$$

Take derivatives:

$$\pi z'_{RS}(p) - \frac{\alpha\eta}{\kappa} z'_{RS}(p) + \frac{y_d}{\kappa} z'_{RS}(p) - \frac{z}{\kappa} z'_{RS}(p) + y_c - p z'_{RS}(p) - z_{RS} z(p) = 0 \tag{49}$$

We have $z'_{RS}(p) = d$. Also, rearranging (48), we have $(p - \pi) = \frac{z-\eta}{d}$. Thus, (49) becomes:

$$-\frac{\alpha\eta}{\kappa} d - \frac{z}{\kappa} d + \frac{y_d d}{\kappa} + y_c - (z - \eta) - z = 0$$

$$z\left(\frac{2\kappa + d}{\kappa}\right) = \eta - \frac{\alpha\eta}{\kappa} d + \frac{y_d d}{\kappa} + y_c$$

This defines the best-response choice of z for each possible value of η. Since z is linear in η, the set of best responses can be implemented by submitting a single linear bid curve $z_D(p)$. To find $z_D(p)$, rearrange for η:

$$\eta = \frac{1}{1 - \frac{\alpha\eta}{\kappa}} \left[\frac{2\kappa + d}{\kappa} z - y_d \frac{d}{\kappa} - y_c\right] \tag{50}$$

Then, substitute (50) into (48):

$$z = (p - \pi)\, d + \frac{2\kappa + d}{\left(1 - \frac{\alpha d}{\kappa}\right)\kappa} z - y_d \frac{d}{\left(1 - \frac{\alpha d}{\kappa}\right)\kappa} - y_c \frac{1}{\left(1 - \frac{\alpha d}{\kappa}\right)}$$

Solving for z, we get the bid curve:

$$z_D(p) = \frac{d y_d + \kappa y_c - d(\kappa - \alpha d)(p - \pi)}{\alpha d + \kappa + d} \tag{51}$$

This proves (24). Taking derivatives of demand,

$$\frac{dz_D}{dp} = \frac{d(\kappa - \alpha d)}{\alpha d + \kappa + d} \tag{52}$$

Thus far, we have taken the slope of residual supply d as given. Since residual supply is negative the sum of $(n-1)$ agents' demand functions, we have:

$$d = -(n-1)\frac{dz_D}{dp} \tag{53}$$

Combining (52) and (53), we get:

$$d = \frac{(n-2)\kappa}{1 + n\alpha}$$

proving (22).

Now, consider (19). From the demand equation (24), residual supply is:

$$z_{RS} = -\sum_{j \neq i} \frac{d y_d + \kappa y_c - d(\kappa - \alpha d)(p - \pi)}{\alpha d + \kappa + d}$$

Hence,

$$\eta = \frac{\sum_{j \neq i} y_d d + \sum_{j \neq i} \kappa y_c}{\alpha d + \kappa + d}$$

Thus, using the projection theorem for Gaussian random variables,

$$E\left[\frac{\theta}{n}\sum_{j \neq i} y_d \mid \eta\right] = \frac{\theta}{n} E\left[\sum_{j \neq i} y_d \mid \eta\right] = \frac{\theta}{n} \frac{Var\left[\sum_{j \neq i} d y_d\right]}{Var\left[\sum_{j \neq i} d y_d\right] + Var\left[\sum_{j \neq i} \kappa y_c\right]} \eta$$

Thus, in order to satisfy,

$$\alpha \equiv \frac{E\left[\frac{\theta}{n}\sum_{j\neq i} y_{jd} \mid \eta\right]}{\eta}$$

we must have

$$\alpha = \frac{d^2\sigma_d^2}{\kappa^2\sigma_c^2 + d^2\sigma_d^2}\frac{\theta}{n}$$

proving (23).

6.4 Proofs for section 4

First, I characterize the distribution of η, when agent i bids against $n-1$ manipulators with contract positions of variance σ_c^2. participating in equilibrium. Sum equilibrium demand across $n-1$ agents to get residual supply:

$$-\sum_{i=1}^{n-1} z_D(p) = (n-2)(p-\pi)\kappa - \frac{1}{n-1}\sum_{i=1}^{n-1} y_{ci}$$

Hence,

$$\eta \equiv -\frac{1}{n-1}\sum_{i=1}^{n-1} y_{ci} \sim N\left(0, \frac{\sigma_c^2}{n-1}\right) \tag{54}$$

Now, we characterize the utility of an agent with demand shocks and forward contract positions, facing an uncertain residual supply curve. Suppose that residual supply is:

$$z_{RSi}(p) = d(p-\pi) + \eta$$

Demand functions are:

$$z_{Di}(p; y_{ci}, y_{di}) = \frac{d}{\kappa+d}y_{di} + \frac{\kappa}{\kappa+d}y_{ci} - (p-\pi)\frac{\kappa d}{\kappa+d}$$

Equate these and solving for p, we have:

$$p - \pi = \frac{dy_{di} + \kappa y_{ci} - \eta(d+\kappa)}{d(d+2\kappa)}$$

Plugging into residual supply, optimal quantity is:

$$z = \frac{dy_{di} + \kappa\eta + \kappa y_{ci}}{d + 2\kappa}$$

Plugging both these expressions into utility:

$$U(z, p) = \pi z + \frac{y_{di} z}{\kappa_i} - \frac{z^2}{2\kappa_i} - pz + py_{ci}$$

Simplifying the result, we have:

$$\frac{(\kappa\eta + dy_{di})^2 - 2\kappa(\eta(d + \kappa) - dy_{di}) y_{ci} + \kappa^2 y_{ci}^2}{2d\kappa(d + 2\kappa)} \tag{55}$$

This is complex, but simplifies if we assume that either $y_{di} = 0$ or $y_{ci} = 0$. First, assume that $y_d = 0$, so agents are pure manipulators. Then (55) becomes:

$$= \frac{\eta^2\kappa^2 - 2\eta\kappa(d + \kappa) y_c + \kappa^2 y_c^2}{2d\kappa(d + 2\kappa)}$$

Now assuming the equilibrium slope of residual supply, $d = (n - 2)\kappa$, this is:

$$\frac{\eta^2 - 2(n - 1)\eta y_c + y_c^2}{2\kappa n(n - 2)}$$

Taking the expectation:

$$E\left[\frac{\eta^2 - 2(n - 1)\eta y_c + y_c^2}{2\kappa n(n - 2)}\right]$$

Now, η is a function of other agents' contract positions, hence is independent of y_{ci}. Moreover, both η and y_{ci} have mean 0. Hence the middle term in the expectation disappears. Hence,

$$E\left[\frac{\eta^2 - 2(n - 1)\eta y_c + y_c^2}{2\kappa n(n - 2)}\right] = E\left[\frac{\eta^2 + y_c^2}{2\kappa n(n - 2)}\right] = \frac{\frac{\sigma_c^2}{n-1} + \sigma_c^2}{2\kappa n(n - 2)} = \frac{\sigma_c^2}{2\kappa(n - 1)(n - 2)}$$

This proves (25).

Now, assume instead that $y_c = 0$, so agents are non-manipulative arbitrageurs. Then

(55) simplifies to:

$$\frac{\eta^2\kappa^2 + 2\kappa d\eta y_{di} + d^2 y_{di}^2}{2d\kappa\left(d + 2\kappa\right)}$$

Agents' expected utility for participating in auction is:

$$E\left[\frac{\eta^2\kappa^2 + 2\kappa d\eta y_{di} + d^2 y_{di}^2}{2d\kappa\left(d + 2\kappa\right)}\right]$$

Again, η is independent of y_{di}, and both have mean 0,so the middle term in the expectation disappears. The expectation thus reduces to:

$$E\left[\frac{\eta^2\kappa^2 + d^2 y_{di}^2}{2d\kappa\left(d + 2\kappa\right)}\right] = \frac{\kappa^2\sigma_\eta^2 + d^2\sigma_d^2}{2d\kappa\left(d + 2\kappa\right)}$$

This proves (30). For the equilibrium expression, substitute $\sigma_\eta^2 = \frac{\sigma_c^2}{n-1}$, $d = (n-2)\,\kappa$ to get:

$$\frac{\kappa^2\frac{\sigma_c^2}{n-1} + (n-2)^2\,\kappa^2\sigma_d^2}{2\,(n-2)\,n\kappa^3} = \frac{\frac{\sigma_c^2}{n-1} + (n-2)^2\,\sigma_d^2}{2\,(n-2)\,n\kappa} = \frac{\sigma_c^2}{2\,(n-2)\,(n-1)\,n\kappa} + \frac{n-2}{2n\kappa}\sigma_d^2$$

This proves (31).

6.5 Proofs for section 5

6.5.1 Proof of proposition 4

If agents with $\kappa_1 \ldots \kappa_k$ have y_{di} and total inventory $\bar{z}$, we solve

$$\max_{z_1 \ldots z_k} \sum_{i=1}^{k} \pi z_i + \frac{y_{di} z_i}{\kappa_i} - \frac{z_i^2}{2\kappa_i}$$

$$\text{s.t.} \sum_{i=1}^{k} z_i = \bar{z}$$

Taking derivatives,

$$\pi + \frac{y_{di}}{\kappa} - \frac{z_i}{\kappa} = \lambda$$

So the solution is

$$\frac{y_{di}}{\kappa_i} - \frac{z_i}{\kappa_i} = \lambda - \pi$$

Or,

$$z_i = y_{di} + \kappa_i \alpha$$

We must have

$$\sum_{i=1}^{k} z_i = \bar{z}$$

Hence,

$$\sum_{i=1}^{k} y_{di} + \kappa_i \alpha = \bar{z}$$

$$\alpha = \frac{\bar{z} - \sum_{i=1}^{k} y_{di}}{\sum_{i=1}^{k} \kappa_i}$$

Plugging into the objective, we have

$$\sum_{i=1}^{k} \pi \left(y_{di} + \kappa_i \frac{\bar{z} - \sum_{i=1}^{k} y_{di}}{\sum_{i=1}^{k} \kappa_i} \right) + \frac{y_{di} \left(y_{di} + \kappa_i \frac{\bar{z} - \sum_{i=1}^{k} y_{di}}{\sum_{i=1}^{k} \kappa_i} \right)}{\kappa_i} - \frac{\left(y_{di} + \kappa_i \frac{\bar{z} - \sum_{i=1}^{k} y_{di}}{\sum_{i=1}^{k} \kappa_i} \right)^2}{2\kappa_i}$$

$$= \pi\bar{z} + \sum_{i=1}^{k} \frac{y_{di}^2}{\kappa_i} + \sum_{i=1}^{k} y_{di} \left(\frac{\bar{z} - \sum_{i=1}^{k} y_{di}}{\sum_{i=1}^{k} \kappa_i} \right) -$$

$$\sum_{i=1}^{k} \frac{y_{di}^2}{2\kappa_i} - \sum_{i=1}^{k} y_{di} \left(\frac{\bar{z} - \sum_{i=1}^{k} y_{di}}{\sum_{i=1}^{k} \kappa_i} \right) - \frac{\sum_{i=1}^{k} \kappa_i^2 \left(\frac{\bar{z} - \sum_{i=1}^{k} y_{di}}{\sum \kappa_i} \right)^2}{2\kappa_i}$$

$$= \pi\bar{z} - \frac{\sum_{i=1}^{k} \kappa_i^2 \left(\frac{\bar{z} - \sum_{i=1}^{k} y_{di}}{\sum_{i=1}^{k} \kappa_i} \right)^2}{2\kappa_i}$$

$$= \pi\bar{z} - \frac{\sum_{i=1}^{k} \kappa_i \left(\bar{z}^2 - 2\bar{z} \sum_{i=1}^{k} y_{di} + \left(\sum_{i=1}^{k} y_{di} \right)^2 \right)}{2 \left(\sum_{i=1}^{k} \kappa_i \right)^2}$$

$$= \pi\bar{z} - \frac{\sum_{i=1}^{k} \kappa_i \bar{z}^2}{2 \sum_{i=1}^{k} \kappa_i} + \frac{\bar{z} \sum_{i=1}^{k} y_{di}}{\sum_{i=1}^{k} \kappa_i} - \frac{\left(\sum_{i=1}^{k} y_{di} \right)^2}{2 \sum_{i=1}^{k} \kappa_i}$$

Now, ignoring the constant term $-\frac{(\sum y_{di})^2}{2\sum \kappa_i}$, utility from holding $\bar{z}$ is:

$$\pi\bar{z} - \frac{\sum_{i=1}^{k} \kappa_i \bar{z}^2}{2\sum_{i=1}^{k} \kappa_i} + \frac{\bar{z}\sum_{i=1}^{k} y_{di}}{\sum_{i=1}^{k} \kappa_i}$$

This is the same as the utility function of a single agent with demand slope $\bar{\kappa} = \sum_{i=1}^{k} \kappa_i$ and demand shock $\bar{y}_d = \sum_{i=1}^{k} y_{di}$, for total quantity $\bar{z}$. The contract position of the cartel is just the sum of all agents' contract holdings, $\bar{y}_c = \sum_{i=1}^{k} y_{ci}$.

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