Competition and Manipulation in Derivative Contract Markets*

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Abstract

This paper studies manipulation in derivative contract markets. When traders hedge factor risk using derivative contracts, traders can manipulate settlement prices by trading the underlying spot goods. In equilibrium, manipulation can make all agents worse off. The model illustrates how contract market manipulation can be defined in a manner distinct from other forms of strategic trading behavior, and how the structure of contract and spot markets affect the size of manipulation-induced market distortions.

Keywords: derivatives, manipulation, regulation

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1 Introduction

This paper studies manipulation in derivative contract markets. If a market participant wants to buy or sell exposure to some risk factor, such as US equities, interest rates, volatility, or energy, the simplest way to do so is often to use a derivative contract. Some examples of derivatives are S&P 500 futures, LIBOR and SOFR derivatives, VIX futures and options, and derivatives for commodities such as corn and wheat, base and precious metals, oil, gas and electricity. Derivatives make up some of the world's largest markets: the total notional size of the interest rate derivatives market alone is over $100 trillion USD.

Derivative contracts are linked to underlying spot markets, either through physical delivery or cash settlement. A cash-settled derivative is simply a contract whose payoff is determined based on some price benchmark, which is constructed based on the trade price of some spot good. If the spot price benchmark accurately reflects some risk factor, contracts settled using the benchmark can be used to trade exposure to this risk factor. For example, a long position in the ICE Houston Ship Channel gas basis futures contract, at settlement, pays its holder some multiple of a gas price index, which is calculated based on prices of physical gas traded in Houston. If contract settlement prices are representative of local gas prices, the futures contract can be used to trade exposure to gas price risk.

If a trader holds positions in both contract and spot markets, she may have incentives to distort spot markets in order to increase contract payoffs. For example, if a trader is long the Houston Ship Channel futures contract, she can increase her contract payoffs by buying physical gas at Houston to raise the contract settlement price. If the trader's futures position is large, her increased futures profits may outweigh any losses incurred by buying physical gas at elevated prices. If many traders bid this way, however, their bids would add noise to the settlement price, creating nonfundamental risk for all agents holding these futures contracts.

Regulators generally consider trading in this manner to constitute illegal manipulation, and regulators have imposed billions of dollars of fines on market participants for manipulation in the past two decades alone. However, contract market manipulation is vaguely defined legally, and poorly understood from the perspective of economic theory. We do not have a precise economic definition of manipulation in contract markets, which differentiates it from other forms of strategic trading behavior. We do not know how manipulation affects the welfare of different classes of market participants, what makes contract markets vulnerable to manipulation, or how to empirically measure contract market manipulation risk.

This paper builds a simple model of contract market manipulation. Within the model, manipulation can be precisely defined, in a way that distinguishes it from other forms of strategic trading behavior. Manipulation can be Pareto-disimproving, harming both hedgers and spot market participants, and policy interventions such as contract position limits can be Pareto-improving. The size of manipulation-induced market distortions depend on the aggregate storage capacity in spot markets, the level of spot market competition, and the size of spot traders' contract positions. The magnitude of manipulation-induced distortions can be estimated using data which is commonly observed by contract market regulators.

In the model, I assume that a number of risk-averse agents have exogenous exposures to a common risk factor. Agents cannot contract on the risk factor directly, but can trade derivative contracts which are tied to the auction price of a spot good. A finite number of spot traders can trade the spot good in an auction, and their marginal value for the good is equal to the risk factor. If spot traders behaved competitively, the spot market would clear with no trade, the spot auction price would be exactly equal to risk factor, and all agents could perfectly share factor risk using derivative contracts.

In strategic equilibrium, however, first-best risk sharing is not attainable. Since spot traders have price impact in the spot market, they have incentives to trade spot goods in a way that increases their derivative contract payoffs. For example, a trader who holds a long contract position has incentives to buy the spot good, to push the spot auction price upwards to increase her contract payoffs. Within the model, I define this form of strategic trading as contract market manipulation.

Manipulation causes auction prices to become noisy signals of the risk factor, creating non-fundamental *basis risk* for all contract holders. Market structure in the spot market determines how vulnerable the contract market is to manipulation. Manipulation-induced distortions are larger when the total storage capacity in spot markets is smaller. Holding fixed aggregate storage capacity, distortions are smaller when spot markets are more competitive; thus, competition policy in spot markets can alleviate manipulation risk, even if it does not affect total storage capacity. Distortions also depend on the size of spot traders' contract positions, so regulatory limits on the size of spot traders' contract positions can decrease manipulation risk.

The possibility of manipulation affects spot traders' contract purchasing decisions. Traders tend to buy less contracts because of manipulation-induced basis risk, but tend to buy more because they anticipate making profits from manipulating spot markets. In equilibrium, spot traders may even over-hedge, purchasing larger derivative positions than their total exposures to the risk factor.

There are two main effects of manipulation on spot traders' welfare. Spot traders receive positive transfers in expectation, because they can move settlement prices in favor of their contract positions on average, but they also face increased risk due to settlement price variance created by other spot traders. The negative effects can be strong enough that spot traders, as a group, are worse off in equilibrium, relative to a world in which all agents behaved competitively. In some settings, a regulator could increase all market participants' welfare by limiting the size of spot traders' contract positions.

The conclusions of the model hold under a number of generalizations: spot traders can have asymmetric holding costs for the spot good, receive arbitrarily distributed inventory shocks for spot goods, and have arbitrarily distributed derivative contract positions. In the general model, manipulation-induced basis risk can be expressed in terms of the slopes of agents' auction bid curves, and the variances and covariances of spot traders' inventory shocks and contract positions. Thus, using the model, basis risk can be estimated using market data which is often observed by regulators. Finally, while the baseline model focuses on cash-settled contracts for expositional simplicity, I show that the analysis applies identically for contracts settled through physical delivery.

Implications for contract market regulators. Regulators currently do not have a precise economic definition of contract market manipulation. In legal proceedings, manipulation is essentially defined as trading with the intent to create artificial prices. Regulators bring cases against market participants based on evidence, often taken from emails and phone calls, that a trader intentionally created price impact for profit. But traders know that their trades move prices; the role that they play in markets is precisely to manage and optimize the price impact of their trades. Defining manipulation as intentional price impact is overly broad: it could in principle cover almost all forms of strategic trading, creating substantial regulatory uncertainty for market participants. But regulators do not currently have a definition which distinguishes contract market manipulation in particular from other legal forms of strategic trading in financial markets.

This paper's framework suggests a way to define contract market manipulation, in a way which distinguishes it from other forms of strategic trading. In the paper's model, price impact has three distinct effects on traders' behavior. First, traders have incentives to provide less liquidity than they would in a competitive market: the slopes of traders' bid curves with respect to prices are lower than they would be in the absence of price impact. Second, traders have incentives to "shade bids", trading less than one spot good for each unit of their inventory shocks. Both of these effects are present in any imperfectly competitive financial market. The third effect, which is unique to contract markets, is that traders who are active in both the contract and spot markets have incentives to trade spot goods in a way that increases payoffs on their contract positions. I propose to define this third effect as illegal contract market manipulation.

Conceptually, there is a simple policy which eliminates traders' incentives to manipulate contract markets, without affecting their incentives for other forms of strategic trading: regulators could disallow spot market participants from holding derivative contracts. Spot traders would then have incentives to reduce liquidity provision and shade bids, but no incentive to manipulate, since they would have no contract positions whose payoffs they can influence. In most settings, banning spot traders from holding derivative positions is likely impractical. However, this hypothetical policy may serve as a useful conceptual benchmark for arguing whether a given pattern of trading behavior is manipulative: regulators could ask whether a given trader's behavior in spot markets could be economically justified as profit-maximizing, if the trader did not hold contract positions. The model also shows that manipulating agents may have incentives to over-hedge, buying more contracts than necessary to hedge factor risk, in order to increase profits from manipulation. This suggests that the buildup of very large contract positions, beyond what appears necessary for hedging factor risks, is suggestive of manipulative intent.

The paper's results also illustrate how regulators can limit manipulation risk by regulating the structure of contract and spot markets. Contract position limits imposed on spot traders can lower manipulation risk, by lowering spot traders' incentives to manipulate. Manipulation-induced distortions will tend to be smaller in spot markets which are more competitive, and in markets with larger aggregate storage capacity for spot goods. In the model of the paper, regulators can quantitatively estimate manipulation-induced basis risk using data which is observed in many markets. These measures could be used by regulators to determine how large contract position limits should be, and whether to approve newly proposed derivative contracts.

Related literature. To my knowledge, this is the first paper to analyze the welfare effects of contract market manipulation, microfounding behavior in both the contract and spot markets. Other theoretical papers studying price manipulation include Goldstein and Guembel (2008), Bond, Goldstein and Prescott (2010), and Bond and Goldstein (2015). These papers study settings in which prices are used as inputs for other decisions, generating incentives for market participants to manipulate prices to distort these decisions. Other theoretical papers on price manipulation include Garbade and Silber (1983), Kyle (1983), Paul (1984), Cita and Lien (1992), Pirrong (2001), Lien and Tse (2006), Allen, Litov and Mei (2006), Kyle (2007), and Guo and Prete (2019).

Two earlier papers study contract market manipulation with cash-settled futures contracts. Kumar and Seppi (1992) analyze the equilibrium of a Kyle (1985) common-valued competitive market-maker model with agents holding futures contracts linked to spot prices of an asset. Dutt and Harris (2005) discuss how to set futures position limits for agents to prevent manipulation from dominating prices. Relative to this literature, this paper endogenizes traders' derivative contract positions, allowing us to analyze the welfare effects of manipulation, and also allows for rich heterogeneity between traders, so the model can realistically be fitted to data.

A number of other papers analyze market and benchmark manipulation empirically. For example, Abrantes-Metz et al. (2012), Gandhi et al. (2015), Bonaldi (2018) analyze LIBOR manipulation, Griffin and Shams (2018) analyze VIX manipulation, Evans et al. (2018) study FX manipulation, Birge et al. (2018) study manipulation in electricity markets, Comerton-Forde and Putniņš (2011) and Comerton-Forde and Putniņš (2014) study stock price manipulation, and Nozari, Pascutti and Tookes (2019) study the related phenomenon of profitable price impact in convertible bond markets. There is also a large regulatory and legal literature on contract market manipulation, which I discuss briefly in subsection 2.2.

Technically, the spot market auction model of this paper builds on the literature on linear-quadratic double auctions; some papers in this literature are Kyle (1989), Vayanos (1999), Vives (2011), Rostek and Weretka (2015), Du and Zhu (2017), Duffie and Zhu (2017), and Lee and Kyle (2018). Some recent papers analyze the question of optimal benchmark design, such as Duffie, Dworczak and Zhu (2017), Duffie and Dworczak (2018), Eisl, Jankowitsch and Subrahmanyam (2017), Coulter, Shapiro and Zimmerman (2018) and Baldauf, Frei and Mollner (2018). While related, this paper's focus is to quantify manipulation risk, rather than to solve for optimal mechanisms for benchmark setting.

Outline. The remainder of the paper proceeds as follows. Section 2 discusses some institutional details of contract market manipulation. Section 3 introduces the baseline model, and section 4 derives the main theoretical results. Section 5 studies various extensions of the model, and shows how to measure manipulation-induced market distortions. Section 6 discusses implications of my findings, and section 7 concludes. Proofs, derivations, and other supplementary material are presented in the online appendix.

2 Institutional background

2.1 Derivative contracts, liquidity mismatch, and manipulation

Derivative contracts are often used by traders to hedge price risks associated with underlying spot goods. Hedging using derivatives is usually preferred to holding spot goods directly, because derivatives are often highly leveraged, and do not have the physical costs associated with transporting and storing spot goods. Derivatives may be cash-settled, in which case they pay their holders based on some price benchmark, or settled by physical delivery, meaning that their holder is entitled to some number of units of the underlying asset.

There is often a *liquidity mismatch* between contract markets and spot markets: contract markets are often much larger than underlying spot markets. For example, the Platts *Inside FERC* Houston Ship Channel benchmark for natural gas prices is based on around 1.4 million MMBtus of natural gas trades per week;[1] open interest in the ICE HSC basis future, which is financially settled based on the Platts benchmark, is more than 75 million MMBtus for many delivery months.[2] The Secured Overnight Financing Rate (SOFR), designed to replace USD LIBOR as an interest rate benchmark, is based on average daily volumes of approximately $1 trillion in overnight treasury-backed repo loans;[3] as of 2014, the total notional volume of contracts linked to USD LIBOR was estimated to be greater than $160 trillion.[4]

There are two main explanations for this liquidity mismatch. The first is speculation: some market participants may wish to trade exposure to price risk simply to express views on the direction of future prices. Derivatives have lower logistical and capital costs to trade than spot goods, so speculators will tend to prefer trading using derivatives. The second is cross-hedging: if prices of a group of spot goods are very correlated, derivative activity often concentrates in one or a few contracts for liquidity reasons. For example, a gas company located in Texas may choose to hedge risks using Henry Hub, Louisiana gas futures contracts, even if Texas gas futures are available, if the Louisiana contract is more liquid.

As a result of this liquidity mismatch, if a spot market participant holds a large derivative position, she may have incentives to trade spot assets non-fundamentally, in a way that increases the payoffs on her contract position. For example, if a gas trader holds a large position in ICE Houston Ship Channel (HSC) basis futures, she can increase her futures payoff by buying spot gas at HSC to increase the benchmark price. The trader may incur losses on her spot market trades, but could generate much larger profits on her futures position.

[1] See "Liquidity in Noth American Monthly Gas Markets" on the Platts website.

[2] ICE Report Center, End of Day reports for HSC basis futures, as of October 24th, 2018. Open interest is above 30,000 contracts for many delivery months, and the contract multiplier is 2,500 MMBtus.

[3] NY Fed's Secured Overnight Financing Rate Data.

[4] Financial Stability Board (2018).

2.2 Legal background

In the US, manipulation and attempted manipulation of contract markets are illegal under the Commodity Exchange Act of 1936. Regulators have policed contract market manipulation aggressively in the last few decades. UBS was fined \$15 million by the CFTC in 2018 for attempting to manipulate gold futures contracts. The CFTC and the FERC have fined traders millions of dollars for manipulating oil, gas, electricity, precious metals, and propane derivatives.[5] Fines for financial derivative manipulation are orders of magnitude larger: banks have been fined over \$10 billion for FX manipulation,[6] over \$8 billion USD for manipulation of LIBOR and other interest rate benchmarks,[7] and over \$500 million for manipulation of the ISDAFIX interest rate swap benchmark.[8]

Manipulation law and policy is a contentious topic. The Commodity Exchange Act outlaws manipulation, but does not define it. The CFTC's operational definition of manipulation essentially states that trades made with the *intent* to create *artificial prices* are manipulative.[9] This definition is still vague, and there has been substantial disagreement in both the economic and legal literatures, both on what can reasonably be defined as manipulation under current law, and on how manipulation should be regulated from a social planner's perspective.

In recent times, the legal literature has largely moved away from the "artificial price" notion, focusing instead on "intent" as the standard of proof for manipulation. Perdue (1987) argues that manipulation should be defined as conduct which "would be uneconomical or irrational, absent an effect on market price." Fischel and Ross (1991), similarly, argue that manipulation can only reasonably be defined based on the intent of the trader.

Regulatory authorities have also largely relied on proof of intent as the primary basis for prosecuting manipulators. Charges are brought based on "smoking gun" evidence that trades were made with the intention of moving benchmark prices. Levine (2014) quotes a number of trader chat messages used in FX manipulation lawsuits. Other

[5] See, for example, CFTC Press Release 6041-11, 128 FERC 61,269, CFTC Press Release 4555-01, CFTC Press Release 7683-18, and CFTC Press Release 5405-07.

[6] Levine (2015)

[7] Ridley and Freifeld (2015)

[8] Leising (2017)

[9] 17 CFR Part 180

examples include the CFTC's lawsuits brought against Parnon Energy, Inc. and others for crude oil manipulation, Energy Transfer Partners, L.P. and others for natural gas manipulation, and Barclays for ISDAFIX manipulation.[10]

Thus, the legal literature and regulatory authorities have largely taken the stance that, under some circumstances, intentional and acknowledged price impact constitutes illegal manipulation. The precise circumstances under which intentional price impact constitutes illegal manipulation, and how manipulation should be distinguished from other forms of strategic trading in financial markets, are unclear. The need to prove intent also imposes a high bar of proof on regulators, as it is difficult to prosecute manipulators in settings where regulators cannot easily access traders' communication records. Legally sophisticated market participants could in principle avoid being implicated in manipulation lawsuits, simply by taking care not to acknowledge the price impact of their trades in recorded media.

There is also some disagreement in the legal and economic literatures as to whether manipulation should be regulated at all. Hieronymus (1977, pg. 328) argues that contract market manipulation will not survive under market competition. Fischel and Ross (1991) argue that "actual trades should not be prohibited as manipulative regardless of the intent of the trader", and that market competition is likely to deter manipulation. Markham (1991) similarly proposes to abandon the concept of manipulation, and to instead empower the CFTC to take a broader set of actions to maintain fair and orderly markets.[11]

3 Model

I assume that agents have exogenous exposures to a common risk factor, and can trade cash-settled derivative contracts to share factor risk. The model has two core frictions. First, agents cannot contract on the risk factor directly: they must instead contract on the auction price of a spot good, whose price is informative about the risk factor. Second, the spot market is imperfectly competitive, so spot good trades move prices. Together,

[10]See the CFTC's website for Parnon Energy, Inc., Energy Transfer Partners, L.P., and Barclays.

[11]A few overviews of market manipulation are Putniņš (2012), Markham (2014), and Putniņš (2020).

these two frictions imply that spot traders who buy contracts to hedge factor risk have incentives to trade strategically in spot markets, to influence the payoffs on their contract positions. Spot traders' strategic trading creates nonfundamental variance in contract settlement prices, and prevents agents from perfectly sharing factor risk.

There are three kinds of agents. There is a representative *pure hedger*, who buys derivative contracts, but cannot trade the spot good. There are also $n > 2$ identical *spot traders*, who can trade both derivative contracts and spot goods. I assume that spot traders are negligibly small relative to the hedger. The hedger and the spot traders are risk-averse with CARA utility over wealth, with identical risk aversions: agent i's utility if she attains wealth W is:

$$U_i(W) = -e^{-\alpha W} \tag{1}$$

There is a risk-neutral competitive *market maker*, who trades derivative contracts, but cannot trade the spot good. The market maker submits a demand schedule, mapping net order quantities, from both the hedger and spot traders, to prices. The market maker cannot distinguish between the hedger's order and spot traders' orders. I will show that, in equilibrium, the market maker will set a constant price for derivatives contracts; let p_c denote the equilibrium contract price.

Agents play a 5-stage game.

1. The hedger and spot traders draw their factor risk exposures, respectively, x_H and $x_1 \dots x_n$.
2. The hedger and spot traders submit market orders, respectively $c_H(x_H)$ and $c_i(x_i)$, to trade derivative contracts with the market maker.
3. The risk factor, ψ, is realized.
4. Spot traders bid to trade the spot good in an auction.
5. Derivative contracts pay all contract holders based on the spot auction price.

In stage 1, the hedger is endowed with x_H units of a productive asset, and each spot trader i is endowed with x_i units of the productive asset, where:

$$x_H \sim N\left(0, \sigma_H^2\right), \; x_i \sim N\left(0, \sigma_x^2\right)$$

A position x_i in the productive asset pays its holder $x_i\psi$ net units of wealth. Thus, the value of each unit of the productive asset depends on a normally distributed *risk factor* ψ, where:

$$\psi \sim N\left(\mu_\psi, \sigma_\psi^2\right)$$

Hence, I refer to x_i as agent i's *factor exposure*. The risk factor ψ may represent, for example, future oil prices, volatility, stock prices, or interest rates; agents may be positively or negatively exposed to each of these sources of risk. I assume that factor exposures are privately observed. I assume the sizes of spot traders' factor risk exposures x_i are comparable with each other, but spot traders' positions are collectively infinitesimal, compared to the hedger's position x_H. For analytical convenience, I normalize wealth by assuming that an agent (spot trader or hedger) who has factor exposure x_i is also endowed with $-x_i\mu_\psi$ units of wealth, so that the expectation of all agents' wealth is 0 regardless of x_i.

In stage 2, agents can trade derivative contracts to hedge their factor exposures. In the baseline model, I assume contracts are cash-settled. This is purely for expositional simplicity: in subsection 5.4, I show that all outcomes are identical if contracts are instead settled by physical delivery of the spot good. I use c_i to denote the contract position of agent i, and I assume that agents' contract positions are private information. I assume it is costless to hold derivative contracts. Derivative contracts are useful because agents can purchase them prior to the realization of ψ, so contracts allow agents to hedge factor risk exposures.

In stage 3, the risk factor ψ is drawn and commonly observed by agents. In stage 4, spot traders bid to trade the spot good in a uniform-price double auction. In stage 5, the spot auction clearing price p is used to settle agents' derivative contracts: that is, each agent is paid $c_i p$. In the spot market, the n spot traders trade a single homogeneous spot good. A spot trader who purchases z_i net units of the spot good attains a monetary payoff:

$$\psi z_i - \frac{1}{2\kappa} z_i^2 \tag{2}$$

The ψz_i component of (2) implies that the marginal value of spot traders for the spot good depends on the risk factor, ψ. The quadratic component, $\frac{1}{2\kappa} z_i^2$, implies that spot traders have decreasing marginal values for the spot good; this can be thought of as a storage or holding cost for the spot good. For physical goods such as oil or gas, the quadratic term might represent physical storage and infrastructure costs; for financial assets such as FX, interest rate swaps, or repo, these costs may correspond to capital or balance sheet costs. Alternatively, these costs may arise from anticipated price impact from liquidating spot positions in the future. The parameter κ can be thought of as a measure of agents' storage capacity for the asset, or the inverse of agents' holding costs. When κ is larger, spot traders are more willing to take on large spot good positions, and in equilibrium the price impact of spot trades will be smaller.

Combining all terms, a spot trader's total wealth, considering her factor exposures and outcomes in the contract and spot markets, is:

$$W_{\text{spot}}\left(x_i, c_i, z_i, p, \psi\right) = \underbrace{\psi x_i - \mu_\psi x_i}_{\text{Factor exposure}} - \underbrace{p_c c_i}_{\text{Contract price}} + \underbrace{p c_i}_{\text{Contract payoff}} + \underbrace{z_i \psi - \frac{1}{2\kappa} z_i^2}_{\text{Spot good payoff}} - \underbrace{p z_i}_{\text{Spot good price}} \tag{3}$$

The pure hedger's total wealth is:

$$W_{\text{hedger}}\left(x_H, c_H, p, \psi\right) = \underbrace{\psi x_H - \mu_\psi x_H}_{\text{Factor exposure}} - \underbrace{p_c c_H}_{\text{Contract price}} + \underbrace{p c_H}_{\text{Contract payoff}} \tag{4}$$

3.1 Benchmarks

To illustrate how the two frictions, noncontractibility and imperfect competition, prevent agents from perfectly sharing risk, we analyze two benchmark cases in which the first-best outcome is attainable: when agents can directly contract on ψ, and when contracts settle to spot market prices, but spot market participants bid competitively.

Contracting directly on ψ. Agents could perfectly share risk if they could purchase contracts tied directly to the risk factor – that is, contracts which pay ψ per contract in stage 5. In the first period, the market maker would set the price of these contracts equal to their expected payoff:

$$p_c = E[\psi] = \mu_\psi$$

The market maker is willing to buy or sell an infinite amount of contracts at this price, since the hedger and spot traders' endowments and contract purchases are uninformative about ψ. An agent with contract position c_i would receive total wealth:

$$\underbrace{x_i \psi - x_i \mu_\psi}_{\text{Factor exposure}} + \underbrace{\psi c_i}_{\text{Contract payoff}} - \underbrace{p_c c_i}_{\text{Contract price}}$$

At price $p_c = \mu_\psi$, spot traders and the hedger would all choose to hedge perfectly, setting $c_i = -x_i$ and $c_H = -x_H$. Thus, all agents would be able to perfectly hedge, transferring all their factor risk exposures to the risk-neutral market maker.

The first friction in the model is that agents cannot contract directly on ψ. This is realistic: in practice, traders cannot contract on abstract risk factors, such as oil prices, volatility, stock prices, or interest rates, directly. Instead, traders can buy and sell contracts which are settled based on price benchmarks set in markets for spot goods, such as the Brent or WTI oil price indices, the VIX, the S&P 500 and Russell 2000, and LIBOR or SOFR.

Competitive spot market bidding. Even if agents cannot contract on ψ directly, first-best risk sharing is possible using contracts settled using spot market prices, if spot traders behaved competitively in the spot market. Each spot trader's marginal value of the spot good is the derivative of (2) with respect to z_i:

$$\psi - \frac{z_i}{\kappa} \tag{5}$$

Suppose each spot trader bid in the auction as if she faced a perfectly elastic residual supply curve; that is, suppose each spot trader i believes that the auction clears at some random price p, which does not depend on i's quantity traded. i then purchases goods up to the point where her marginal utility is equal to the auction price; this is implemented by submitting a bid curve equal to the inverse of (5):

$$z_{Bi}(p) = \kappa(\psi - p) \tag{6}$$

The spot auction would then always clear at price $p = \psi$. There would be no trade of the spot good; this is efficient, since traders' valuations for the spot good are identical. Anticipating that $p = \psi$, spot traders and the hedger would buy contracts to fully hedge their factor exposures, so agents would be able to perfectly share factor risk. In this benchmark case, the auction plays no allocative role: its role is to elicit spot traders' marginal utility for the asset, which is equal to ψ, and then using this as a benchmark to settle derivative contracts.

The second friction in the model is that the spot market is imperfectly competitive. Spot traders are strategic, and recognize that their spot good trades move the settlement price p. Suppose a spot trader has a negative factor exposure x_i, and buys a positive contract position c_i to hedge her factor risk exposure; in the spot market, the trader has an incentive to increase her bid in the spot auction, in order to increase p and thus her contract payoffs. This makes p a noisy signal of ψ, and prevents perfect risk sharing. I analyze how these incentives play out in equilibrium in the following section.

3.2 Discussion of model assumptions

Assumptions on the contract market. As in Kyle (1985), I assume that there is a risk-neutral market maker, and that spot traders' factor risk exposures, and thus their contract purchases, are infinitesimal relative to the hedger. These assumptions imply that market makers will set the price of contracts equal to μ_ψ, the expectation of the risk factor ψ. Intuitively, since spot traders are infinitesimal, their contract purchases are perfectly hidden within the representative hedger's order flow, and thus net order flow is totally uninformative about the settlement price p. The assumption that spot traders are small, so contract purchases have no price impact, is likely a reasonable approximation for many (but not all) contract markets: subsection 2.1 shows that contract markets are often much larger than spot markets.

I assume that there is no cost to hold contract positions. This is a reasonable approximation in many settings: derivatives are useful because they are highly leveraged financial assets, so they have no physical holding costs, and low capital costs relative to spot goods. Agents' contract positions are still limited, however, by their ability to bear factor risk: an agent who purchases a very large contract position holds a very large exposure to ψ. This is why, in the model, spot traders will not have incentives to build up infinitely large contract positions.

Contract purchases may be motivated by speculation rather than hedging. In appendix B.4, I assume agents have heterogeneous beliefs about the mean μ_ψ of the risk factor; this leads agents to purchase nonzero contract positions, because agents think contracts are mispriced in the first period. The implications for manipulation in the spot market are unchanged.

Finally, while the model of the contract market is quite stylized, assumptions about the contract market only matter for the results in section 4, regarding spot traders' contract purchasing decisions and welfare. As I discuss in section 5, spot traders' contract positions at settlement are sufficient statistics for estimating their incentives to manipulate spot markets. Details of why and how spot traders entered into their contract positions do not affect these calculations. Thus, my measure for manipulation-induced welfare losses for hedgers are valid in a variety of different models of the contract market.

Assumptions on the spot market. The baseline spot market model is intentionally stylized, in order to illustrate the main forces at work in the model. In section 5, I explore various extensions, allowing spot traders to have heterogeneous spot good storage capacities, inventory shocks, and arbitrarily distributed contract positions.

Throughout the paper, I take the number of spot market participants, n, and their spot good holding capacities, κ, as exogeneous; I am essentially analyzing contract market manipulation holding fixed spot market structure. Spot market structure can change, as agents enter, exit, and adjust their holding costs, but these changes tend to be costly and take time. For physical spot goods, such as oil and gas, holding costs depend on costly infrastructure such as pipelines and storage facilities; for financial assets, many benchmarks are set in inter-dealer markets, which do not allow free entry. Thus, the assumption that spot market structure is essentially fixed in the short run seems reasonable in many settings; however, the metrics I propose would need to be updated over time, as spot market structure changes.

A related concern is that, if the contract market is very vulnerable to manipulation, market participants could hedge factor risk by holding spot goods directly. However, as subsection 2.1 discusses, market participants would generally prefer to hedge using derivatives, because holding spot goods involves higher logistical and capital costs. If manipulation makes derivative contracts sufficiently unattractive that market participants choose to hedge using spot goods, despite their higher costs, there is a social welfare loss, because market participants could share risk at lower cost using derivatives in the first-best outcome.

I assume that the representative hedger cannot trade the spot good. It is equivalent to assume that the hedger can trade the spot good, but has no storage capacity: effectively, the hedger has $\kappa = 0$. This implies that the hedger has perfectly inelastic demand for the spot good, and cannot adjust her purchases in response to prices. For example, an airline or chemicals plant may regularly purchase oil or other commodities as inputs to production, but these inputs may be very inelastic in the short run. This implies that the hedger must submit a perfectly inelastic demand curve in the spot market, and has no ability to manipulate by adjusting her spot trades depending on her contract position.

I assume agents optimize independently of each other; however, in the model, spot traders have incentives to collude. A spot trader who increases settlement prices generates profits for all other traders with long contract positions; thus, colluding spot traders, who coordinate their spot market bids to maximize joint profits, would manipulate more per unit contract than independently optimizing traders. This is an interesting direction to explore, but I leave this to future work.

4 Equilibrium

Proposition 1 describes equilibrium values of spot traders' bid curves, auction prices, contract purchases, and expected utility for all agents.

Proposition 1. *For any $\alpha, \mu_\psi, \sigma_\psi^2, \kappa, \sigma_x^2, \sigma_H^2, n$, there is a unique equilibrium, in which spot traders' spot market bids and contract purchasing strategies are linear and symmetric across agents. In stage 1, the market maker sets contract price $p_c = \mu_\psi$. Spot traders' bid curves in the spot market are:*

$$z_{Bi}(p; c_i, \psi) = \frac{1}{n-1} c_i - \frac{n-2}{n-1} \kappa (p - \psi) \tag{7}$$

Spot auction prices are:

$$p - \psi = \frac{\sum_{i=1}^{n} c_i}{n(n-2)\kappa} \sim N\left(0, \frac{\sigma_c^2}{n(n-2)^2 \kappa^2}\right) \tag{8}$$

Spot traders' contract positions $c_i(x_i)$ *are linear in traders' factor exposures* x_i*, so spot traders' contract positions are normally distributed with mean 0 and variance* σ_c^2*, where:*

$$c_i(x_i) = -t x_i,\ \sigma_c^2 = t^2 \sigma_x^2 \tag{9}$$

where t *satisfies:*

$$t \equiv \left(1 + \frac{\alpha\sigma_\eta^2 - \kappa}{\left(\alpha\sigma_\psi^2 \kappa\right)\left((n^2 - 2n)\kappa + \alpha\sigma_\eta^2\right)}\right)^{-1} \tag{10}$$

and σ_η^2 *is the unique positive value satisfying:*

$$\sigma_\eta^2 = \frac{\sigma_x^2}{n-1}\left(1 + \frac{\alpha\sigma_\eta^2 - \kappa}{\left(\alpha\sigma_\psi^2 \kappa\right)\left((n^2 - 2n)\kappa + \alpha\sigma_\eta^2\right)}\right)^{-2} \tag{11}$$

$$\sigma_\eta^2 > \frac{\kappa - \left(\alpha\sigma_\psi^2\right)(n^2 - 2n)\kappa^2}{\alpha\left(1 + \alpha\sigma_\psi^2 \kappa\right)} \tag{12}$$

Spot traders' expected utility, as a function of t*, is:*

$$-\frac{\sqrt{\frac{(n^2-2n)\kappa}{\alpha\left(\frac{t^2\sigma_x^2}{n-1}\right)+(n^2-2n)\kappa}}}{\sqrt{1 - \alpha\sigma_x^2\left(\alpha\sigma_\psi^2(1-t)^2 + \left(\frac{\alpha\left(\frac{t^2\sigma_x^2}{n-1}\right)-\kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left((n^2-2n)\kappa+\alpha\left(\frac{t^2\sigma_x^2}{n-1}\right)\right)}\right)t^2\right)}} \tag{13}$$

The hedger's net contract position is linear in the hedger's factor exposure x_H*:*

$$c_H\left(x_H\right)=-\frac{\sigma_\psi^2}{\sigma_\psi^2+Var\left(p-\psi\right)}x_H=-\frac{\sigma_\psi^2}{\sigma_\psi^2+\frac{\sigma_c^2}{n(n-2)^2\kappa^2}}x_H \tag{14}$$

The hedger's wealth has mean 0, and its variance, over uncertainty in ψ, x_H, *and all* x_i*'s, is:*

$$Var\left(W_{hedger}\left(x_H,c_H,p,\psi\right)\right)=\frac{Var\left(p-\psi\right)}{Var\left(p-\psi\right)+\sigma_\psi^2}\sigma_\psi^2\sigma_H^2=\left(\frac{\frac{\sigma_c^2}{n(n-2)^2\kappa^2}}{\frac{\sigma_c^2}{n(n-2)^2\kappa^2}+\sigma_\psi^2}\right)\sigma_\psi^2\sigma_H^2 \tag{15}$$

4.1 Spot market distortions and manipulation-induced basis risk

Spot traders' equilibrium bids, (7), differ from their competitive bids, (6), in two ways. The first difference, which is well-known in the double-auctions literature, is that traders decrease liquidity provision due to their price impact. The slope of (7) with respect to prices is lower than the slope of agents' competitive bids, (6), by a factor $\frac{n-2}{n-1}$. The second difference is that traders' equilibrium bid curves depend on their contract positions c_i, even though contract positions do not affect traders' marginal value for spot goods. Spot traders hold contracts with payoffs that depend on p, and traders can move p by trading the spot good, so traders have incentives to trade the spot good to increase contract payoffs. Within the model, I define the second force as manipulation: it is a distortion which is unique to traders who are active in both spot and contract markets, and it is distinct from price impact avoidance, which is present more generally in any market where traders have price impact.

Expression (7) shows that increasing a trader's contract position by 1 unit causes her to increase her spot good bid curve by $\frac{1}{n-1}$ units: as n increases, contract positions affect bids less. Intuitively, when the spot good auction is competitive, agents need to buy more of the spot good to move prices a given amount. Thus, the cost of manipulation is higher, so spot traders manipulate less per unit contract that they hold. In the limit as n grows large, expression (7) converges towards (6), and traders' contract positions have no effect on their spot good bids.

Since auction prices depend on spot traders' contract positions c_i, which depend on traders' random factor exposures x_i, the equilibrium auction price p is a noisy signal of the risk factor ψ. The difference, $p-\psi$, is normally distributed, and expression (8) characterizes its variance:

$$Var\left(p-\psi\right)=\frac{\sigma_c^2}{n\left(n-2\right)^2\kappa^2} \tag{16}$$

I call this variance *manipulation-induced basis risk*.

Comparative statics of basis risk. We will define three quantities that influence the level of basis risk. First, define the aggregate *storage capacity* of the market as $n\kappa$, the sum over all n spot traders' storage capacities. $n\kappa$ would be the slope of aggregate market demand for the spot good, if all agents submitted bid curves as if they had no price impact, as in (6). Second, define the level of market *competition* simply as n, the number of spot traders present in the market. Third, define the size of contract positions as σ_c^2, the variance of an individual spot trader's contract position. This is an equilibrium object in the model, but in practice it can often be observed directly by regulators, and directly influenced using policies such as contract position limits.

Basis risk decreases to 0 as storage capacity becomes infinitely large, $n\kappa\to\infty$, holding fixed n and σ_c^2. Intuitively, when $n\kappa$ is large, the price impact of manipulative spot trading decreases towards 0, causing prices converge to ψ. However, while *prices* become competitive, spot market *quantities* do not become competitive. From (7), the coefficient on contract positions in agents' bids is $\frac{1}{n-1}$, independent of κ. If κ increases holding n fixed, prices converge to ψ, but spot market bids, and thus allocations in the spot market, are still distorted in the limit.

Basis risk also decreases towards 0 as markets become competitive, $n\to\infty$, holding fixed $n\kappa$ and σ_c^2. This is because, from (7), the coefficient on c_i in agents' equilibrium bids is $\frac{1}{n-1}$. In more competitive markets, agents' contract positions affect their bidding decisions less, and thus create less price variance, even if aggregate storage capacity is fixed. This implies that competition policy in spot markets is an effective tool for limiting manipulation risk: blocking mergers, and splitting large spot traders into smaller entities, can decrease manipulation-induced basis risk, even if it does not increase the market's total storage capacity for the spot good.[12] Note that, in contrast to the case of increasing storage capacity, increasing competition also causes equilibrium spot market bids to converge to competitive bids, so spot market trade quantities also converge to their efficient values.

Basis risk also increases in the size of spot traders' contract positions, σ_c^2, holding fixed n and $n\kappa$. Intuitively, spot traders manipulate more, and create more basis risk, when their contract positions are larger. This implies that regulatory position limits imposed on spot traders can lower manipulation-induced basis risk.

4.2 Spot traders' contract positions

Expression (9) of proposition 1 shows that, in the unique equilibrium of the model, agents' optimal contract purchases c_i are linear in their factor exposures x_i. The coefficient of proportionality, defined as t in expression (10), describes how aggressively agents are hedging; that is, how many contracts agents purchase per unit of their factor exposures.

If all traders bid competitively in spot markets, traders would perfectly hedge, setting $t = 1$. In equilibrium, t can be greater or smaller than 1. On the one hand, spot traders anticipate that others will manipulate, making spot prices noisy signals of ψ, decreasing their incentives to buy contracts to hedge factor risk. On the other hand, spot traders anticipate that they can profit on average from their contract positions, since they can move spot prices in their favor, increasing their incentives to buy contracts.

The second force may dominate the first, so we can have $t > 1$ in equilibrium. In this case, spot traders will "over-hedge", buying contract positions which are larger than their original factor risk exposures x_i. In other words, agents may purchase contract positions so large that they actually increase their exposures to factor risk, because of anticipated profits from moving contract settlement prices in their favor. Agents do not purchase infinitely large contract positions, because the increased exposure to basis risk eventually

[12] Note that σ_c^2 is the variance of an individual trader's contract position. Thus, as $n \to \infty$, the size of aggregate contract positions across spot traders, $\sum_i c_i$, also becomes unboundedly large. However, the variance of $\sum_i c_i$ grows at rate n, whereas the coefficient on c_i in agents' equilibrium bids is $\frac{1}{n-1}$, contributing a factor $\frac{1}{n^2}$ to the asymptotic behavior of price variance. As a result, manipulation-induced price variance is declining in n at asymptotic rate $\frac{1}{n}$, holding fixed σ_c^2 and $n\kappa$.

overwhelms any anticipated profits from manipulation.

From expression (10), t is greater than 1 if:

$$\kappa > \alpha\sigma_\eta^2$$

where σ_η^2, from (11), is the expected variance of residual supply in the spot market. Intuitively, spot traders have incentives to buy large contract positions if storage capacity κ is large, relative to risk aversion α and the variance of residual supply σ_η^2. Appendix B.2 shows that the amount of over-hedging in equilibrium can be unboundedly large: there exist parameter settings which make the equilibrium value of t arbitrarily large. Appendix B.3 shows that increasing n, holding other parameters fixed, causes t to converge to 1 at rate $\frac{1}{n^2}$: when there are a large number of market participants, spot traders hedge perfectly, and the market approaches the first-best outcome.

4.3 Spot trader welfare

Expression (13) shows spot traders' expected welfare, over all sources of uncertainty in the model. This allows us to assess whether spot traders are better or worse off in equilibrium, relative to the competitive-bidding benchmark. Expression (13) also allows us to calculate welfare for values of t other than its equilibrium value. This corresponds to welfare in a limited social planner's problem, in which the planner can force all spot traders to buy contracts according to some prespecified value of t, but cannot influence traders' behavior in spot markets. This can be thought of as a reduced-form model of various actions contract market regulators can take to limit the size of traders' contract positions, such as imposing contract position limits.[13]

The three panels of figure 1 show spot traders' welfare as a function of t, alongside the equilibrium and competitive values of t and spot trader welfare, for three different sets of input parameters. A given spot trader's manipulation generates both positive

[13] I do not directly analyze position limits because my model requires agents' contract positions to be Gaussian, so bounds on the size of agent's contract positions would be intractable. In the context of the model, appendix B.5 shows that the planner could implement any positive value of t in equilibrium by imposing quadratic taxes or subsidies on spot traders' contract positions, charging spot traders kc_i^2 for buying c_i contracts.

and negative externalities on other spot traders, so a number of different outcomes are possible: spot traders may gain or lose on average from manipulation, the equilibrium t may be greater or smaller than 1, and spot traders may hedge more or less than is optimal for spot traders' welfare.

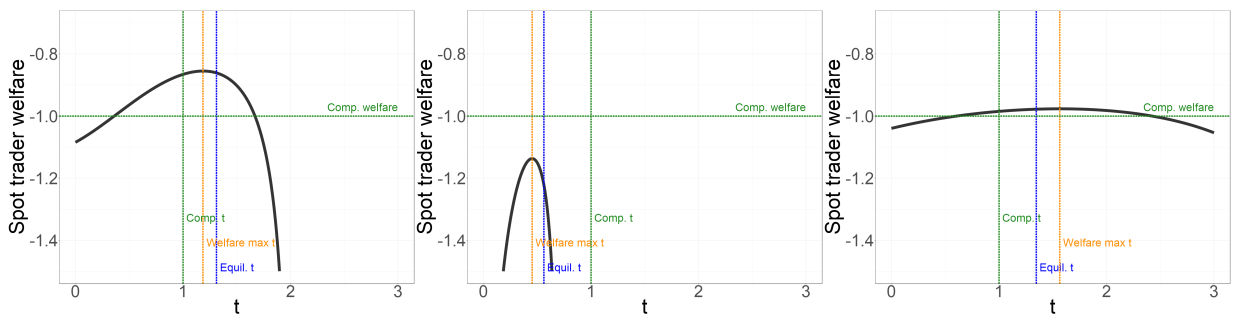
In the left panel, manipulation increases spot traders' welfare, relative to the competitive benchmark. Spot traders also over-hedge, choosing $t > 1$ in equilibrium. However, spot traders would do even better as a group if they could commit to holding smaller contract positions. Spot traders face a kind of prisoner's dilemma: manipulation by one spot trader creates a negative externality on other spot traders by increasing basis risk, so all spot traders would prefer a lower value of t than the equilibrium value. A lower value of t would also benefit the hedger, since it would decrease basis risk, so a regulator could create a Pareto improvement by limiting the size of spot traders' contract positions.

In the middle panel, like the left panel, spot traders would prefer for t to be smaller than its equilibrium value. In addition, spot traders' equilibrium welfare is below the competitive value of -1. As the following subsection shows, the hedger is always worse off in equilibrium relative to the competitive benchmark. Thus, in this example, manipulation is Pareto disimproving relative to the competitive benchmark, decreasing welfare for both spot traders and the hedger. Intuitively, this is because spot traders' losses from manipulation-induced basis risk outweigh their gains from extracting a transfer from the hedger in expectation.

In the right panel, spot traders would actually prefer for t to be higher than its equilibrium value. This is because manipulation by one spot trader actually creates a positive externality on other spot traders in spot markets: manipulators make non-fundamental trades in spot markets, so other spot traders profit as market makers, buying low and selling high in the spot auction. In this case, spot traders face a kind of coordination problem: all spot traders would be better off if each trader bought more contracts and manipulated more. This would, however, increase basis risk and decrease the hedger's welfare, so an increase in t can never be Pareto-improving.

Together, these examples show that manipulation can be Pareto-dominated relative to the competitive benchmark, and regulatory intervention to decrease spot traders' hedging aggressiveness can be Pareto improving. Appendix B.1 shows additional comparative

Figure 1: Spot trader welfare



Notes. Expected welfare of spot traders, (13), as a function of the hedging aggressiveness parameter t, for three different sets of parameters. The green vertical and horizontal lines denote the competitive values of t and welfare, which are always equal to 1 and -1 respectively. The blue line denotes the unique equilibrium value of t, and the orange line denotes the value of t which maximizes spot traders' welfare. The parameters for the left panel are $n = 3, \alpha = 1, \kappa = 0.5, \sigma_\psi^2 = 0.3, \sigma_x^2 = 0.5$; for the middle panel, they are $n = 5, \alpha = 1, \kappa = 0.05, \sigma_\psi^2 = 0.9, \sigma_x^2 = 1$; for the right panel, they are $n = 3, \alpha = 1, \kappa = 0.8, \sigma_\psi^2 = 1.5, \sigma_x^2 = 0.05$.

statics of the model, illustrating how equilibrium outcomes are affected by different model primitives.

4.4 Hedger welfare

Expression (14) shows how manipulation affects the representative hedger's contract purchasing decisions. The hedger buys less contracts per unit of her risk exposure, because under manipulation, contracts are noisier hedges for factor risk. The optimal hedge is to buy

$$-\frac{\sigma_\psi^2}{\sigma_\psi^2 + Var\,(p - \psi)} \tag{17}$$

units of contracts, per unit of her factor risk exposure. Intuitively, (17) is the coefficient from regressing the risk factor ψ on the auction price p. With manipulation-induced basis risk, p is a noisier signal of ψ, so the hedger optimally hedges less per unit of her factor risk exposure. As a result, the hedger's wealth has variance:

$$\frac{Var(p-\psi)}{Var(p-\psi)+\sigma_\psi^2}\sigma_\psi^2\sigma_H^2 \tag{18}$$

The ratio $\frac{Var(p-\psi)}{Var(p-\psi)+\sigma_\psi^2}$ is equal to $1-R^2$, where R^2 is the coefficient of determination from regressing p on ψ. When basis risk is 0, $R^2=1$, there is no basis risk, so the hedger can perfectly hedge and face no wealth uncertainty. As basis risk becomes very large, $R^2=0$, so the hedger buys no contracts, and is fully exposed to factor risk. Thus, in the model, fixing σ_ψ^2 and σ_H^2, manipulation-induced basis risk, $Var(p-\psi)$, is a sufficient statistic for the hedger's welfare losses from manipulation.

5 Extensions

In this section, I study a number of extensions of the baseline model. In subsection 5.1, I allow spot traders to have asymmetric storage capacities κ_i, as well as asymmetric distributions of contract positions. In subsection 5.2, I allow spot traders to have inventory shocks. In subsection 5.3, I show how a regulator could measure manipulation risk empirically. Throughout these subsections, I restrict attention to stages 4 and 5 of the game: I study spot traders' bidding behavior in the spot market, taking spot traders' contract positions as exogenous random variables, rather than studying the full multi-stage game.[14] Finally, subsection 5.4 shows that the results of the model also apply to physical delivery contracts.

[14]The multi-stage game with asymmetric agents is difficult to solve, because each trader faces a different variance of residual supply in the spot market, $\sigma_{\eta i}^2$. Solving for equilibrium would involve finding a set of bids for all agents, such that agents' bids are best responses given spot market bids of other agents, and the variances $\sigma_{\eta i}^2$, and then bidding and contract purchasing decisions are consistent with all residual supply variances $\sigma_{\eta i}^2$. This cannot be solved in closed-form, and appears not to add much insight compared to studying the spot market game directly, taking contract positions as exogeneous.

5.1 Asymmetric agents

Suppose that the wealth of spot market participant i, if she buys z_i net units of the spot good at price p, is:

$$W_{spot,i}(c_i, z_i, p) = \underbrace{pc_i}_{\text{Contract payoff}} + \underbrace{\psi z_i - \frac{z_i^2}{2\kappa_i}}_{\text{Spot good payoff}} - \underbrace{pz_i}_{\text{Spot good price}} \tag{19}$$

Expression (19) generalizes (3) from the baseline model in two ways. First, (19) allows spot market participants to have different holding capacities, κ_i, for the spot good. Agents with larger κ_i have more elastic demand for the spot good; these may be agents who have more storage space for physical spot goods, or lower capital costs for trading financial spot goods. Second, I assume that contract positions c_i have mean 0 and full support, but otherwise can be arbitrarily distributed: in particular, contract positions need not be identically distributed across agents, and different agents' contract positions can be arbitrarily correlated.[15]

Compared to (3), I also omit factor exposure and contract price terms from (19). These terms do not depend on z_i, so they do not affect traders' decisions in the spot market. Also, in the spot market, agents' bids are ex-post best responses over uncertainty in other agents' bids, so agents' optimization problem is nonstochastic, and we can solve (19) without assuming that agents have CARA utility over wealth.

Expression (19) is an accurate model of spot market behavior under many different assumptions about how the contract market works: for example, as appendix B.4 discusses, spot traders may buy contracts to speculate rather than hedge, or as appendix B.5 discusses, spot traders' contract purchases may have price impact. A spot trader who holds a long contract position profits from higher spot prices, regardless of why she entered into the contract in the first place. Thus, spot traders' contract positions are an

[15]Full support is needed only to justify the ex-post equilibrium concept: when contract positions have full support, the intercepts of the residual supply curves faced by each trader also have full support, so spot traders' bid curves are fully determined by requiring them to be best responses for all possible realizations of residual supply. The assumption that each c_i has mean 0 is not needed to solve the model, but implies that $p - \psi$ has mean 0; without it, p would not be equal to ψ in expectation, but the model conclusions would otherwise be unchanged.

observable sufficient statistic for spot traders' incentives to manipulate spot markets.

The following proposition characterizes spot good bids and prices in the unique equilibrium of the general model.

Proposition 2. *When spot traders' wealth is described by (19), there is a unique linear ex-post equilibrium, in which* i *submits the bid curve:*

$$z_{Bi}(p; c_i) = \frac{b_i}{\sum_{j \neq i} b_j} c_i - b_i (p - \psi) \tag{20}$$

The spot auction price is:

$$p - \psi = \frac{1}{\sum_{i=1}^{n} b_i} \sum_{i=1}^{n} \left[\frac{b_i}{\sum_{j \neq i} b_j} c_i \right] \tag{21}$$

bid slopes b_i *satisfy:*

$$b_i = \frac{B + 2\kappa_i - \sqrt{B^2 + 4\kappa_i^2}}{2} \tag{22}$$

and $B \equiv \sum_{i=1}^{n} b_i$ *is the unique positive solution to the equation:*

$$B = \sum_{i=1}^{n} \frac{2\kappa_i + B - \sqrt{B^2 + 4\kappa_i^2}}{2} \tag{23}$$

Manipulation incentives depend most directly on the coefficient on c_i, $\frac{b_i}{\sum_{j \neq i} b_j}$, in expression (20). From (22), b_i is increasing in κ_i, so agents with larger storage capacities have greater incentives to manipulate, in the sense that they trade more of the spot good per unit contract that they hold. If the market is fairly competitive, we can obtain a simple approximation to agents' optimal bid curves. Define agent i's *capacity share* s_i as:

$$s_i \equiv \frac{\kappa_i}{\sum_{i=1}^{n} \kappa_i} \tag{24}$$

Since κ_i represents the spot good storage capacity of agent i, s_i can be thought of as i's share of total storage capacity in the spot market. Define s_{max} as the largest agent's capacity share:

$$s_{max} \equiv \max_i s_i$$

The following proposition derives bounds on the c_i coefficients, $\frac{b_i}{\sum_{j \neq i} b_j}$, as functions of s_i and s_{max}.

Claim 1. When $s_{max} < \frac{1}{2}$, we have:

$$s_i \leqslant \frac{b_i}{\sum_{j \neq i} b_j} \leqslant \left(1 + \frac{s_{max}}{1 - 2s_{max}}\right) s_i \tag{25}$$

Claim 1 implies that the coefficient on c_i in i's bid in equilibrium is within a factor

$$\left(1 + \frac{s_{max}}{1 - 2s_{max}}\right)$$

of her capacity share, s_i. Thus, when s_{max} is small, c_i coefficients are approximately equal to capacity shares. Intuitively, when markets are relatively competitive, an agent who constitutes roughly 10% of the market's storage capacity will increase her spot market bids by approximately 10% of the size of her contract position. The factor s_i thus generalizes the factor $\frac{1}{n-1}$ in (7), the expression for spot market bid curves in the baseline model.

If all agents' contract positions have equal variance, σ_c^2, we can also derive bounds for manipulation-induced basis risk. Define the capacity HHI (Herfindahl-Hirschman index) as:

$$HHI \equiv \sum_{i=1}^{n} s_i^2 \tag{26}$$

The HHI is a standard measure of market concentration used in antitrust analysis. When agents are symmetric, $\kappa_i = \kappa$, the HHI is simply $\frac{1}{n}$.

Claim 2. When $s_{max} < \frac{1}{2}$, agents' contract positions have the same variance σ_c^2, and agents' contract positions are independently distributed, manipulation-induced basis risk satisfies:

$$\frac{\sigma_c^2}{B^2} HHI \leqslant Var\left(p - \psi\right) \leqslant \left(1 + \frac{s_{max}}{1 - 2s_{max}}\right)^2 \frac{\sigma_c^2}{B^2} HHI \tag{27}$$

Using claim 2, we can generalize the comparative statics discussion in subsection 4.1. In the asymmetric model, we can use the HHI as an inverse measure of market competition. Analogous to the baseline model, we can define the aggregate *storage capacity* as $\sum_i \kappa_i$. The ratio between aggregate storage capacity and the aggregate bid slope, B, can be bounded by a finite constant factor, which approaches 1 when s_{max} approaches 0. To see this, from claim 12 of appendix D, we have:

$$\left(1 - \frac{s_{max}}{1 - s_{max}}\right) \kappa_i \leqslant b_i \leqslant \kappa_i$$

for all i. Thus, we have:

$$\left(1 - \frac{s_{max}}{1 - s_{max}}\right) \sum_{i=1}^{n} \kappa_i \leqslant B \leqslant \sum_{i=1}^{n} \kappa_i$$

This shows that, as aggregate storage capacity $\sum_{i=1}^{n} \kappa_i$ increases towards ∞, as long as $s_{max} < \frac{1}{2}$, the slope of aggregate market demand B must also increase towards ∞. Claim 2 then implies that, analogous to the baseline model, basis risk decreases to 0 as the aggregate storage capacity $\sum_i \kappa_i$ rises to ∞ (and thus B increases to ∞), holding HHI and σ_c^2 fixed; as HHI decreases towards 0, holding $\sum_i \kappa_i$ and σ_c^2 fixed; and as σ_c^2 decreases, holding fixed HHI and $\sum_i \kappa_i$.

Next, suppose traders' contract positions are independently distributed, but suppose the variance of contract positions differs across agents. Let σ_{ci}^2 represent the variance of trader i's contract position. Basis risk is then:

$$Var\left(p - \psi\right) = \frac{1}{B^2} \sum_{i=1}^{n} \left(\frac{b_i}{\sum_{j \neq i} b_j}\right)^2 \sigma_{ci}^2 \tag{28}$$

Expression (28) states that basis risk is the weighted sum of contract position variances, σ_{ci}^2, with weights $\left(\frac{b_i}{\sum_{j \neq i} b_j}\right)^2$. These weights are larger for traders with larger storage capacity κ_i. Intuitively, agents with larger spot good storage capacity will trade more in the spot market per unit contract they hold, so basis risk will be higher when agents with large storage capacity hold large contract positions. This suggests that, in order to lower manipulation-induced basis risk, it may be beneficial to impose stricter position limits on market participants with larger storage capacity for the spot good.

5.2 Inventory shocks

Next, we add inventory shocks to the model. Assume spot traders' wealth is:

$$W_{\text{spot},i}(c_i, z_i, y_i, p) = \underbrace{pc_i}_{\text{Contract payoff}} + \underbrace{\psi z_i - \frac{(z_i + y_i)^2}{2\kappa_i}}_{\text{Spot good payoff}} - \underbrace{pz_i}_{\text{Spot good price}} \tag{29}$$

Expression (29) assumes that agents enter the spot market with some existing inventory position, y_i, in the spot good.[16] I assume that each y_i has mean 0 and full support, but otherwise can be arbitrarily distributed; inventory shocks can be asymmetrically distributed across agents, and can be arbitrarily correlated with other agents' inventory shocks and contract positions. Inventory shocks imply that agents will trade the spot good even if they do not hold contract positions, so that not all trade volume in the spot market is caused by manipulation.

Proposition 3. *When spot traders' wealth is described by (29), there is a unique linear ex-post equilibrium in the general model, in which* i *submits the bid curve:*

$$z_{Bi}(p; y_i, c_i) = -\frac{b_i}{\kappa_i} y_i + \frac{b_i}{\sum_{j \neq i} b_j} c_i - b_i (p - \psi) \tag{30}$$

The spot auction price is:

$$p - \psi = \frac{1}{\sum_{i=1}^n b_i} \sum_{i=1}^n \left[-\frac{b_i}{\kappa_i} y_i + \frac{b_i}{\sum_{j \neq i} b_j} c_i \right] \tag{31}$$

bid slopes b_i *satisfy:*

$$b_i = \frac{B + 2\kappa_i - \sqrt{B^2 + 4\kappa_i^2}}{2} \tag{32}$$

[16] Inventory shocks y_i can alternatively be interpreted as preference shocks for the spot good: expanding the quadratic term in (29), we have:

$$-\frac{z^2}{2\kappa_i} - \frac{y_i z}{\kappa_i} - \frac{y_i^2}{2\kappa_i}$$

Ignoring the constant $\frac{y_i^2}{2\kappa_i}$ term, y_i simply linearly shifts i's marginal value of the spot good.

and $B \equiv \sum_{i=1}^{n} b_i$ *is the unique positive solution to the equation:*

$$B = \sum_{i=1}^{n} \frac{2\kappa_i + B - \sqrt{B^2 + 4\kappa_i^2}}{2} \tag{33}$$

First, I specialize proposition 3 to the case in which agents are fully symmetric.

Corollary 1. *Suppose that* $\kappa_i = \kappa$, $Var(y_i) = \sigma_y^2$, $Var(c_i) = \sigma_c^2$ *for all* i, *and all* y_i *and* c_i *are independent. Spot traders' equilibrium bids are:*

$$z_{Bi}(p; y_i, c_i) = -\frac{n-2}{n-1} y_i + \frac{1}{n-1} c_i - \frac{n-2}{n-1} \kappa (p - \psi) \tag{34}$$

Price variance is:

$$Var(p - \psi) = \frac{\sigma_y^2}{n\kappa^2} + \frac{\sigma_c^2}{n(n-2)^2 \kappa^2} \tag{35}$$

Expressions (34) and (35) are very similar to the expressions for bids and prices in the baseline model, in proposition 1; the only difference is the inclusion of y_i terms. The coefficient on y_i in equilibrium bid curves, (34), is $\frac{n-2}{n-1}$. As is known in the double-auctions literature, price impact causes agents to "shade bids", passing through their inventory shocks less than one-to-one into bids. As n increases, the y_i coefficient increases towards 1, whereas the c_i coefficient decreases towards 0. Intuitively, y_i reflects a fundamental, inventory- or utility-driven component of bids, whereas c_i reflects a nonfundamental component, reflecting agents' preferences over prices, driven by their contract positions. When n is higher and agents have less price impact, their bids reflect fundamentals more, and preferences over prices less. The model thus shows how contract-linked bidding can be distinguished from bid shading.

Another implication of (34) is that, in relatively competitive markets, the coefficient on c_i tends to be much smaller than the coefficient on y_i: an agent who receives a unit inventory shock changes her spot bid curve by approximately one unit, whereas an agent with a unit contract shock changes her spot bid by approximately $\frac{1}{n}$ units. Thus, when n is large, most of the variation in agents' bids (and thus settlement prices) will be driven by inventory shocks, even if the variance of c_i is much larger than the variance of y_i. Quantitatively, from (35), inventory positions and contract positions contribute equal amounts to settlement price variance when:

$$\sigma_c = (n-2)\,\sigma_y \tag{36}$$

that is, when the standard deviation of contract positions is $(n-2)$ times greater than the standard deviation of inventory shocks. As subsection 2.1 discusses, contract markets are often much larger than spot markets in practice. It is a puzzle why this liquidity mismatch can be sustained, without creating very large manipulation incentives for spot traders. Expression (36) provides a simple answer: contract positions can be much larger than inventory positions, without contract-driven trading dominating settlement price variance, as long as spot markets are sufficiently competitive.

Expression (36) could be used as a simple rule-of-thumb for evaluating whether a given market is vulnerable to manipulation: regulators need only check whether the total size of spot traders' outstanding contract positions is more than $(n-2)$ times larger than the total size of their inventory positions. This test only formally works under strong symmetry assumptions, but may be a reasonable first-pass when limited market data are available.

When agents have different κ_i values, from (30) of proposition 3 and (22) of proposition 2, agents with higher values of κ_i – larger capacity relative to the market – will submit bid curves with higher p_i coefficients, lower y_i coefficients, and higher c_i coefficients.[17] Intuitively, these agents submit bid curves which are steeper with respect to prices, so they absorb more of other agents' inventory shocks. They shade their inventory positions more, trading less per unit of their inventory shocks y_i. They manipulate more, trading more per unit of their contract positions c_i. As a result, expression (31) shows that equilibrium prices depend on a weighted sum of c_i and y_i: the weights on y_i are lower, and the weights on c_i are higher, for agents with larger κ_i values.

5.3 Measurement

Next, I show how manipulation-induced basis risk could be empirically estimated by a regulator. To ensure that basis risk is finite, we assume that all y_i and c_i have finite

[17] I formally show that the coefficient on y_i, $\frac{b_i}{\kappa_i}$, is decreasing in κ_i, in appendix D.1.2.

variances and covariances. Expression (31) shows that auction prices are linear in agents' inventory and contract positions: this implies that the moments of auction prices are functions of the moments of y_i and c_i. As in proposition 3, we define $B \equiv \sum_{i=1}^{n} b_i$ as the sum of agents' bid slopes b_i. Define the coefficient vectors $\mathbf{k_c}, \mathbf{k_y}$ as:

$$\mathbf{k_c} = \begin{pmatrix} \frac{b_1}{\sum_{j \neq 1} b_j} \\ \vdots \\ \frac{b_n}{\sum_{j \neq n} b_j} \end{pmatrix}, \mathbf{k_y} = \begin{pmatrix} \frac{b_1}{\kappa_1} \\ \vdots \\ \frac{b_n}{\kappa_n} \end{pmatrix}$$

Define the covariance matrices of agents' contract position, $\mathbf{\Sigma_{cc}}$, agents' inventory positions, $\mathbf{\Sigma_{yy}}$, and the inventory-contract covariance matrix $\mathbf{\Sigma_{yc}}$, respectively, as matrices with elements:

$$\mathbf{\Sigma_{cc}}(i,j) = Cov\left(c_i, c_j\right), \ \mathbf{\Sigma_{yy}}(i,j) = Cov\left(y_i, y_j\right), \ \mathbf{\Sigma_{yc}}(i,j) = Cov\left(y_i, c_j\right)$$

Proposition 4. *In the general model of proposition 3, basis risk is:*

$$Var(p - \psi) = \frac{1}{B^2}\left(\mathbf{k'_y \Sigma_{yy} k_y} - 2\mathbf{k'_y \Sigma_{yc} k_c} + \mathbf{k'_c \Sigma_{cc} k_c}\right) \tag{37}$$

Proposition 4 shows that, in order to estimate basis risk, regulators need to observe data on agents' bid slopes in spot markets, and the variances and covariances of agents' inventory and contract positions.

Bid slopes. In order to estimate $\mathbf{k_c}$ and $\mathbf{k_y}$, the regulator must estimate b_i for all agents, that is, the slope of each agent's equilibrium bid curve with respect to prices. The coefficient vectors $\mathbf{k_c}$ and $\mathbf{k_y}$ are functions of both b_i and agents' spot holding capacity κ_i; however, κ_i can be recovered from the vector of b_i values, and the aggregate bid slope B, by inverting (32) of proposition 3. For benchmarks which are set in auctions, bid slopes could be estimated directly from agents' auctions bids. If bidding data are not available, regulators could instead model or estimate agents' spot good holding capacities, κ_i, and then calculate implied bid slopes b_i by solving for equilibrium in proposition 3.

Inventory-contract variance and covariances. The regulator must also estimate the variance-covariance matrices of c_i and y_i. In many markets, regulators have access to detailed data on agents' positions in both spot and derivative markets, so the variance and covariance matrices could be estimated using historical sample moments of c_i and y_i.[18] If c_i is not observable, the regulator could derive bounds on $\mathbf{\Sigma_{cc}}$ based on contract position limits which are imposed by exchanges or regulators. If y_i is not observable, under some distributional assumptions, the regulator could estimate or bound $\mathbf{\Sigma_{yy}}$ using the observed volume of trade, since more variable inventory shocks imply higher trade volumes in spot markets.

Using Proposition 4, regulators could estimate model-predicted basis risk for any given contract market. Regulators could, for example, set a maximum allowable level of basis risk, and set contract position limits low enough that (37) does not exceed this level. For newly proposed contracts, regulators could also use (37) to estimate manipulation-induced basis risk, and regulators could reject new contracts, or require changes to contract specifications, if predicted manipulation-induced basis risk is too high.

In appendix C.5, I define another measure of manipulation-induced distortions, *manipulation rents*, which are equal to the expected profits that spot traders as a group earn, due to their ability to move spot market prices. I show how to estimate manipulation rents in the general asymmetric model. In the model of the paper, basis risk is a sufficient statistic for the hedger's welfare losses from manipulation; however, manipulation rents may also be a useful metric to quantify spot traders' aggregate potential profits from manipulation.

5.4 Physical delivery contracts

Thus far, we have assumed all contracts are cash-settled. In this subsection, based on an equivalence result in Kyle (2007), I show that the analysis is essentially unchanged if contracts are instead settled by physical delivery. Suppose that, instead of cash settlement, each unit of the derivative contract entitles agents to receive a unit of the spot good during the settlement auction. All agents – the hedger, the market maker, and the n spot traders – submit bid curves in the spot auction, taking into account the quantities that

[18] The covariance matrices are still high-dimensional, so finite-sample error may be a concern; this can be solved using a variety of econometric methods, such as low-dimensional factor models for agents' contract and inventory covariances.

they are entitled to from their contract positions. Market clearing in the spot market thus requires the sum of the quantities of the spot good that each agent takes delivery of – consisting of promised units from agents' contract positions, plus the amount that agents bid for in the spot market – to sum to 0. Let:

$$c_H, z_{B,H}(p), c_{MM}, z_{B,MM}(p)$$

respectively denote contract positions and spot market bids from the hedger, H, and the market maker, MM. Let q_H be the net amount of the spot good that the hedger takes delivery of, summing over her contract position and her spot good trades at settlement:

$$q_H \equiv c_H + z_H$$

and let q_{MM} be defined likewise. We have assumed that the hedger and market maker cannot make or take delivery of the spot good, since they have no storage capacity. Hence, q_H and q_{MM} must be equal to 0, so the hedger and the market maker are forced to submit market orders – perfectly inelastic demand curves – to sell exactly the amount of the spot good that they are entitled to in contracts:

$$z_{B,H}(p) = -c_H, \ z_{B,MM}(p) = -c_{MM} \tag{38}$$

In practice, (38) can be interpreted as saying that the hedger and the market maker exit their contract positions just before settlement. The market maker's wealth is thus:

$$W_{MM}(c_{MM}) = \underbrace{-p_c c_{MM}}_{\text{Contract price}} - \underbrace{p(-c_{MM})}_{\text{Spot good sales}} \tag{39}$$

As in the baseline model, (39) implies that the market maker purchases contracts at p_c, and each unit of the contract effectively pays out the spot price p. Thus, risk-neutral competitive market makers will set $p_c = E[p \mid c_{MM}]$, as in the baseline model. The hedger's wealth is:

$$W_{\text{hedger}}(x_H, c_H, p, \psi) = \underbrace{\psi x_H - \mu_\psi x_i}_{\text{Factor exposure}} - \underbrace{p_c c_H}_{\text{Contract price}} - \underbrace{p(-c_H)}_{\text{Spot good sales}} \tag{40}$$

Expression (40) is identical to (4) in section 3, so physical delivery contracts and cash-settled contracts are identical from the perspective of the hedger. If $p = \psi$, these contracts are effective for hedging, because the hedger buys contracts at p_c, sells the resultant inventory of the spot good at ψ, and does not actually take delivery of any of the spot good. Effectively, the hedger hedges by pre-purchasing a promise to deliver at μ_ψ, and then selling the promised units at the realized price ψ. If $p \neq \psi$, the hedger hedges imperfectly, because the price that the hedger gets from liquidating her spot inventory does not reflect ψ perfectly.

Spot traders. With physical delivery contracts, spot traders essentially begin spot market trading endowed with c_i units of the spot good. Spot trader i's wealth, if she purchases an additional z_i units of the spot good, is thus:

$$W_{\text{spot}}(x_i, c_i, z_i, p, \psi) =$$

$$\underbrace{\psi x_i - \mu_\psi x_i}_{\text{Factor exposure}} - \underbrace{p_c c_i}_{\text{Contract price}} + \underbrace{(z_i + c_i)\psi - \frac{1}{2\kappa}(z_i + c_i)^2}_{\text{Spot good payoff}} - \underbrace{p z_i}_{\text{Spot good price}}$$

Expanding, and ignoring terms that do not depend on c_i or z_i, we have:

$$c_i\psi + z_i\psi - \frac{1}{2\kappa}\left(z_i^2 + 2z_i c_i + c_i^2\right) - p z_i \tag{41}$$

Expression (38) shows that the hedger and the market maker make and take no deliveries of the spot good. Thus, in order for spot markets to clear, the total quantities that spot traders take deliveries of – which are equal to the quantities z_i purchased in the spot market, plus traders' total contract positions c_i – must sum to 0:

$$\sum_{i=1}^{n} z_i(p; c_i, \psi) + c_i = 0$$

As above, we can define the quantity spot trader i takes delivery of, q_i, as the sum of z_i and c_i:

$$q_i(p; c_i, \psi) \equiv z_i(p; c_i, \psi) + c_i \tag{42}$$

Thus, when $z_i = -c_i$, trader i does not take any deliveries: she closes all her contract positions just before settlement. When $z_i = 0$, trader i takes delivery of her entire contract position c_i. When z_i is between 0 and $-c_i$, i closes some fraction of her contract position, and takes delivery of the remainder.

Proposition 5 describes outcomes in the spot market in a competitive-bidding benchmark, and proposition 6 describes behavior in strategic equilibrium.

Proposition 5. *With physical delivery contracts, if spot traders behave competitively, as if residual supply in the spot market is perfectly elastic, spot traders' bids are:*

$$z_{Bi}(p; c_i, \psi) = -c_i + \kappa(p - \psi) \tag{43}$$

Spot traders' aggregate demand for deliveries of the spot good is:

$$\sum_{i=1}^{n} q_i(p; c_i, \psi) = \sum_{i=1}^{n} z_{Bi}(p; c_i, \psi) + c_i = \sum_{i=1}^{n} \kappa(p - \psi) \tag{44}$$

Hence, the spot market clears with no trade: spot traders do not make or take delivery of the spot good, and the market clearing price is $p = \psi$.

Proposition 6. *In strategic equilibrium with physical delivery contracts, spot traders' optimal bids in the spot market are:*

$$z_{Bi}(p; c_i, \psi) = -\frac{n-2}{n-1} c_i + \frac{n-2}{n-1} \kappa(p - \psi) \tag{45}$$

The quantity of the spot good that traders take delivery of, q_i, *is identical to spot bids in the baseline model, (7) of proposition 1:*

$$q_{Bi}(p; c_i, \psi) = \frac{1}{n-1} c_i - \frac{n-2}{n-1} \kappa(p - \psi) \tag{46}$$

The market clearing price is identical to (8) of proposition 1 in the baseline model:

$$p - \psi = \frac{\sum_{i=1}^{n} c_i}{n(n-2)\kappa} \sim N\left(0, \frac{\sigma_c^2}{n(n-2)^2 \kappa^2}\right) \tag{47}$$

Propositions 5 and 6 show that physical delivery and cash-settled contracts behave identically within the model. Proposition 5 shows that, if spot traders behave competitively, ignoring their price impact, there are no deliveries in the spot market: all spot traders set $z_i = -c_i$, which corresponds to closing their entire contract positions just before settlement. Spot prices are equal to the risk factor ψ, so the hedger can perfectly hedge.

In strategic equilibrium, (46) of proposition 6 shows that the net quantity that spot traders take delivery of, under physical delivery contracts, is identical to (7) in the cash-settlement case. From (47), prices are identical to what they are under cash settlement, so all welfare implications of both models for all agents are identical.

The intuition behind this equivalence, which is explained in Kyle (2007), is that cash settlement and physical delivery contracts are exactly equivalent if traders have access to "market-on-expiration" orders: that is, if traders can buy or sell arbitrary quantities of the spot good at exactly the cash settlement price. This equivalence holds in the auction model of this paper. A trader who holds a physical delivery contract, and thus is entitled to c_i units of the spot good, can sell $z_i = -c_i$ units, thus taking delivery of no net units of of the spot good, and receiving a monetary payment of $c_i p$, effectively converting her physical delivery contract position into a cash-settled position. Similarly, a cash-settled contract holder can submit an order for $z_i = c_i$ units, converting her cash-settled contract into a physical delivery contract.[19]

Proposition 5 shows that there are some differences in how manipulation occurs notationally, under cash-settled versus physical delivery contracts. For cash-settled contracts, a long contract holder manipulates by buying the spot good at settlement. For physical delivery contracts, comparing (43) and (45), contract holders manipulate by setting z_i smaller in absolute value than $-c_i$: that is, a long contract holder closes less of her contract positions than she would in a competitive world, taking delivery on some

[19] Kyle (2007) also discusses conditions under which this equivalence may not hold. For some price benchmarks, exact "market-on-expiration" orders are difficult or impossible – for example, when benchmarks are calculated as time-weighted average prices. One other form of manipulation, which is only possible with physical-delivery contracts, is that a long manipulator may aim to buy up enough of the underlying asset that it is essentially impossible for shorts to fulfill their delivery requirements. The manipulator then uses the threat of default to extract large payments from shorts to close out contracts. In cases like this, payments from shorts may depend not only on spot market conditions, but also their perceived costs of defaulting on contract obligations, so the model in this paper may not fully capture market outcomes in these cases.

fraction of her contracts. In both cases, spot traders manipulate by making or taking deliveries of spot goods, changing spot prices and thus payoffs on contract positions.

6 Discussion

6.1 Implications for contract market regulation

This paper's framework helps regulators to precisely define contract market manipulation, and to distinguish it from other forms of strategic trading in imperfectly competitive markets. The results show how structural policy tools, such as position limits, can be used to reduce market participants' incentives to manipulate.

Defining manipulation, and behavioral solutions. As subsection 2.2 discusses, contract market manipulation is illegal, but the law does not precisely define manipulation. Courts, regulators and academics have faced substantial difficulty in precisely defining manipulation, and this is a major barrier to enforcing manipulation law. This paper presents a potential definition of contract market manipulation. When strategic spot market traders hold derivative contracts linked to spot markets, traders have incentives to modify their trading behavior, to increase the payoffs on their contract positions. Within the model, this force can be distinguished from two other incentives traders in imperfect markets have to distort trading behavior: decreasing liquidity provision, that is, lowering the slopes of their bid curves relative to a competitive benchmark; and "shading bids", that is, lowering the extent to which inventory shocks affect bids.

If spot traders were barred from holding derivative contracts, they would decrease liquidity provision and shade bids, but would not have incentives to "manipulate" in the sense of trading spot goods to increase contract payoffs. In most markets, it is likely impractical to bar spot traders from holding derivative positions. However, the thought experiment of eliminating spot traders' derivative positions may be useful as a conceptual benchmark for arguing that illegal contract market manipulation is occuring: regulators might ask whether a given pattern of spot good trades could have been profitable, if a trader did not hold derivative contracts.

This definition also shows what trading patterns regulators might look for, to demonstrate that market participants are engaging in illegal manipulation. Trading in spot markets, which could not be profitable on a standalone basis, may be profitable once payoffs from contract positions are taken into account. The model also shows that traders may "over-hedge" in equilibrium, buying contract positions much larger than their factor exposures, deliberately building up exposure to factor risk, since they know they can manipulate contract settlement prices.

Some of the features in the model may be difficult for regulators to bring to data. For example, while inventory or preference shocks are precisely defined within the model, these may be difficult to estimate in practice. Market participants who are accused of manipulation could thus always claim that their spot trades are driven by inventory or preference shocks, rather than the desire to manipulate. The paper's results suggest that over-hedging is symptomatic of manipulation, but demonstrating that agents are over-hedging requires estimating the size of market participants' factor risk exposures, which may be difficult if market participants have positions in many different products with correlated values. Moreover, market participants may enter into positions to speculate rather than hedge risk, and a market participant could always argue that a position which appears to be an over-hedge, with manipulative intent, was in fact simply a speculative bet on factor risk.

Structural regulation. Regulators also use a variety of structural interventions in contract markets to limit manipulation risk. This paper's results assist regulators in these efforts by shedding light on the forces that affect manipulation risk. Manipulation risk depends on the size of spot traders' contract positions. Regulators could impose position limits on spot traders' contract positions, especially close to contract settlement. Position limits are currently imposed in many derivative contract markets, but their primary purpose seems to be preventing excessive price volatility.[20] As a result, the CFTC currently applies position limits more harshly for pure financial market participants than spot traders.[21] The results of this paper suggest that, for the purposes of alleviating

[20] The CFTC's stated purpose for speculative position limits is to "protect futures markets from excessive speculation that can cause unreasonable or unwarranted price fluctuations"; hence position limits do not exist solely to combat manipulation, although according to my theory they can be an effective tool for doing so.

[21] Contract position limits do not apply for market participants who have *bona fide* commercial risks to hedge; see the CFTC's website on Speculative Limits.

manipulation risk, position limits should actually be more harsh for spot traders, since they have a greater ability to trade spot goods to influence settlement prices.

Spot market storage capacity and competitiveness also affect manipulation risk.[22] This suggests, for example, that a recent regulatory effort to move interest rate benchmarks to larger and more competitive underlying markets could potentially alleviate manipulation risk.[23] Moreover, this implies that regulators cannot determine manipulation incentives by focusing on derivative contract markets in isolation. Contract market regulators should monitor storage capacity and concentration in spot markets, perhaps in collaboration with antitrust and spot market regulators, as spot market structure is an important factor affecting contract market manipulation risk.

Subsection 5.3 of the paper shows how to estimate the size of manipulation-induced market distortions, using data which regulators observe in many markets. Regulators could use these measures to decide whether to approve new contracts, and how large contract position limits should be. These measures would also inform regulators how changes in spot market structure, such as mergers or changes in aggregate storage capacity, would influence manipulation risk.

6.2 Mechanisms for benchmark setting

This paper uses uniform-price double auctions as a reduced-form model of price benchmarks. This is potentially a reasonable model for many, but not all, benchmarks. Some benchmarks, such as VIX and the LBMA gold price, are determined using actual auctions. Derivative contracts for many equity indices are also settled based on exchange opening or closing auction prices.[24] Some benchmark-setting mechanisms may produce outcomes similar to uniform-price double auctions. The WM/Reuters FX fixing[25] and the ISDAFIX interest rate swap benchmark[26] (now the ICE swap rate) are set using exchange prices over

[22]Regulators already regulate benchmark setting: principles for financial benchmarks have been released by the International Organization of Securities Commissions (IOSCO (2013)), the FCA began regulating a number of benchmarks, and then in 2018 EU law was revised to include benchmark regulation, along similar principles to the IOSCO report.

[23]See, for example, Duffie and Stein (2015),

[24]Understanding the Special Opening Quotation (SOQ)

[25]WM/Reuters FX Benchmarks

[26]ICE Swap Rate

short time periods; these outcomes may be reasonably well-modelled by uniform-price double auctions. Likewise, some benchmarks for commodities such as oil and gas are set using volume-weighted average prices in specific geographical locations, over relatively short time spans; if the underlying goods are relatively homogeneous, uniform-price auctions may be a reasonable model of outcomes.[27]

Other benchmarks are less well approximated by auctions, and the results of this paper may apply less well in these settings. Some benchmarks are based on trades of underlying assets in markets with large search or transportation frictions. For example, the CME Feeder Cattle Index is based on US-wide cattle trade prices; the price of cattle traded in New York on any given day may differ substantially from the price of cattle traded in California. Other markets are organized as core-periphery networks, with central dealers trading with peripheral counterparties (Wang, 2017; Duffie and Wang, 2016). In these markets, agents' manipulation incentives may depend on their physical location, or their position in the dealer network, in addition to their holding capacity for spot goods.

Some benchmarks are not based on prices of verifiable trades, but rely on market participants to self-report trades or potential trades. For example, LIBOR is based on banks' announcements of their borrowing costs,[28] and some natural gas benchmarks are based on market participants' reports of their trades.[29] In these settings, market participants can manipulate benchmarks simply by falsely reporting trades. This is essentially a form of fraud, and is qualitatively different from the kind of contract market manipulation studied in this paper. The results of the paper thus do not fully capture the ways in which market participants can distort contract payoffs in these settings.

7 Conclusion

The regulation of contract market manipulation is a contentious topic in both academic and policy circles. Illegal manipulation is essentially defined as trading with the intent

[27]Klemperer and Meyer (1989) shows that supply function competition between dealers produces outcomes equivalent to uniform-price double auctions.

[28]ICE LIBOR

[29]CFTC Press Release 5409-07

to move prices. This definition is broad and vague, and it is unclear that regulators and courts currently apply it in a way that can improve market quality and social welfare.

This paper develops a simple model of contract market manipulation. In the model, manipulation is a market failure, which can cause equilibrium outcomes to be Pareto dominated. Regulatory intervention, such as imposing position limits on spot traders, can alleviate this market failure, and in some cases can improve the welfare of all market participants. The model admits a precise definition of contract market manipulation, distinguishing it from other forms of strategic trading in financial markets. The model also illustrates how market primitives affect the likelihood of manipulation, and how different policy tools can be used to limit manipulation risk. The results of the paper can thus potentially be used to assist policymakers in their efforts to detect and regulate manipulation in derivative contract markets.

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Online Appendix

A Proof of proposition 1

Repeating (3), spot traders' total wealth can be written as:

$$W_{\text{spot}}\left(x_i, c_i, z_i, p, \psi\right) = \underbrace{\psi x_i - \mu_\psi x_i}_{\text{Factor exposure}} - \underbrace{\mu_\psi c_i}_{\text{Contract price}} + \underbrace{p c_i}_{\text{Contract payoff}} + \underbrace{z_i \psi - \frac{1}{2\kappa} z_i^2}_{\text{Spot good payoff}} - \underbrace{p z_i}_{\text{Spot good price}} \tag{48}$$

A.1 Spot market bidding

To solve the spot market auction, I adopt the standard solution concept of equilibrium in ex-post optimal bid curves. A bid curve is ex-post optimal if it is optimal for any realization of other agents' bid curves which occurs in equilibrium. I further restrict attention to linear bid curves, and strategies which are symmetric across agents.

From the perspective of agent i, the spot auction defines a *residual supply curve*, $z_{RSi}(p)$, specifying the number of units of the underlying asset that i is able to trade at price p. This is the negative of the sum of all other agents' bid curves:

$$z_{RSi}(p) = -\sum_{j \neq i} z_{Bj}\left(p; c_j, \psi\right) \tag{49}$$

In equilibrium, if bid curves are linear, residual supply functions will also be linear, with a fixed slope:

$$z_{RSi}(p) = d(p - \psi) + \eta_i \tag{50}$$

Where the random intercept η_i depends on uncertainty in other traders' bid curves, resulting from uncertainty in their contract positions c_i. I solve the spot auction model using the standard Kyle (1989) trick: I assume agents can choose the quantity they want to purchase for every possible realization of η_i, then show that these choices can be implemented by an affine demand schedule.

Claim 3. In the spot market, given d, agents' optimal bid curves are:

$$z_{Bi}(p; c_i, \psi) = \frac{\kappa}{d+\kappa} c_i - \frac{\kappa d}{d+\kappa}(p-\psi) \tag{51}$$

Proof. Spot trader i's wealth is given by (48). Trader i chooses her bid curves after x_i and ψ are realized, so I analyze agents' choices conditional on x_i and ψ. Assume the agent faces a residual supply curve as described in (50), and rearrange to get the inverse residual supply function:

$$p_{RS}(z_i; \eta_i, \psi) = \psi + \frac{z_i - \eta_i}{d} \tag{52}$$

Suppose i can condition her purchase decision on η_i. We can write (48) as:

$$W = \psi x_i - \psi \mu_x + p_{RS}(z_i; \eta_i, \psi) c_i - \mu_x c_i + z_i \psi - \frac{1}{2\kappa} z_i^2 - z_i p_{RS}(z_i; \eta_i, \psi) \tag{53}$$

All components of (53) are known to i, so i simply chooses her purchase quantity z_i to maximize (53). Differentiate with respect to z_i:

$$p'_{RS}(z_i; \eta_i, \psi) c_i + \psi - z_i p'_{RS}(z_i; \eta_i, \psi) - p(z_i; \eta_i, \psi) - \frac{z_i}{\kappa} = 0 \tag{54}$$

From (52), we have $p'_{RS}(z_i; \eta_i) = \frac{1}{d}$, hence (54) becomes:

$$\frac{c_i}{d} + \psi - \frac{z_i}{d} - p_{RS}(z_i; \eta_i, \psi) - \frac{z_i}{\kappa} = 0 \tag{55}$$

Expression (55) implicitly defines the optimal choice of z_i given η_i. Expression (55) defines an affine bid curve; solving for z_i, we attain expression (51). Since (51) passes through exactly all pairs (z_i, p) which are i's optimal choices for some realization of η_i, i can do no better than submitting bid curve (51). $\square$

Now, note that from (49), the slope of residual supply d facing any given agent is equal to the sum of all $n-1$ other agents' bid slopes. Thus, in equilibrium, we must have:

$$d = (n-1)\frac{\kappa d}{d+\kappa}$$

Solving for d, we get:

$$d = (n-2)\,\kappa \tag{56}$$

Plugging this into (51), we get (7) of proposition 1. This implies that agents' optimal bids are:

$$z_{Bi}\,(p; c_i, \psi) = \frac{1}{n-1} c_i - \frac{n-2}{n-1} \kappa\,(p-\psi)$$

proving (7).

To get prices, sum bids and add to 0:

$$\sum_{i=1}^{n} \frac{1}{n-1} c_i - \frac{n-2}{n-1} \kappa\,(p-\psi) = 0$$

Solving for $p-\psi$, and using that agents' contract positions are normally distributed with variance σ_c^2, we get (8).

A.2 Spot trader welfare conditional on x_i

Now, I analyze how much utility trader i achieves in expectation, if she has factor exposure x_i. Suppose an agent with factor exposure x_i is bidding against a residual supply curve of the form (50).

Claim 4. Agent i's expected utility given $\alpha, \sigma_\psi^2, \kappa, x_i, c_i, \sigma_\eta^2, d$ is:

$$\sqrt{\frac{d^2+2\kappa d}{\alpha\kappa\sigma_\eta^2 + d^2 + 2\kappa d}} \left(-\exp\left(-\frac{\alpha}{2} \left(-\alpha\sigma_\psi^2\,(c_i+x_i)^2 - \frac{\alpha\sigma_\eta^2 - \kappa}{\alpha\kappa\sigma_\eta^2 + d^2 + 2\kappa d} c_i^2 \right)\right)\right) \tag{57}$$

Proof. To calculate expected utility over uncertainty in η_i and ψ, we first write expected utility from the auction as a function of η_i, fixing c_i. Rearranging residual supply from (50), we have:

$$p = \psi + \frac{z_i + \eta_i}{d} \tag{58}$$

Wealth is:

$$W = \psi x_i - \mu_\psi x_i - \mu_\psi c_i + p c_i + z_i \psi - \frac{1}{2\kappa} z_i^2 - z_i p$$

Plugging in (58) for prices and rearranging, we have:

$$W = \psi x_i - \mu_\psi x_i - \mu_\psi c_i + \psi c_i + \frac{\eta_i c_i}{d} + \\ \frac{z_i(\eta_i; c_i)\, c_i}{d} - \frac{(z_i(\eta_i; c_i))^2}{2\kappa} - \frac{(z_i(\eta_i; c_i))^2}{d} - \frac{\eta_i z_i(\eta_i; c_i)}{d} \quad (59)$$

Now, to find an expression for $z_i(\eta_i; c_i)$, we eliminate prices from expression (51) for optimal bid curves and expression (50) for residual supply, to get z_i as a function of η_i:

$$z_i(\eta_i; c_i) = \frac{\kappa}{d + 2\kappa}(c_i - \eta_i) \quad (60)$$

Plugging (60) into expression (59) for wealth, and simplifying, we have that wealth is:

$$\psi x_i - \mu_\psi x_i - \mu_\psi c_i + \psi c_i + \frac{\eta_i c_i}{d} + \frac{(c_i - \eta_i)^2 \kappa}{2d^2 + 4\kappa d}$$

Given our assumption of CARA utility, agents' utility is:

$$-\exp\left(-\alpha\left(\psi x_i - \mu_\psi x_i - \mu_\psi c_i + \psi c_i + \frac{\eta_i c_i}{d} + \frac{(c_i - \eta_i)^2 \kappa}{2d^2 + 4\kappa d}\right)\right) \quad (61)$$

We first integrate (61) over uncertainty in η_i, assuming that η_i is normally distributed with mean 0 and variance σ_η^2, to get:

$$-\sqrt{\frac{d^2 + 2\kappa d}{\alpha\kappa\sigma_\eta^2 + d^2 + 2\kappa d}} \exp\Bigg[\\ -\alpha\left(\psi x_i - \mu_\psi x_i + \psi c_i - \mu_\psi c_i\right) + \frac{\alpha}{2}\left(\frac{\alpha\sigma_\eta^2 - \kappa}{\alpha\kappa\sigma_\eta^2 + d^2 + 2\kappa d}\right) c_i^2\Bigg] \quad (62)$$

This gives expected utility over uncertainty in η_i, conditional on ψ. Now, we integrate (62) over uncertainty in ψ, which is normally distributed with mean μ_ψ and variance σ_ψ^2, to get:

$$\sqrt{\frac{d^2+2\kappa d}{\alpha\kappa\sigma_\eta^2+d^2+2\kappa d}}\left(-\exp\left(-\frac{\alpha}{2}\left(-\alpha\sigma_\psi^2\left(c_i+x_i\right)^2-\frac{\alpha\sigma_\eta^2-\kappa}{\alpha\kappa\sigma_\eta^2+d^2+2\kappa d}c_i^2\right)\right)\right)$$

as desired. □

A.3 Optimal hedging

Using claim 4, we can find spot traders' optimal choice of c_i. We conjecture that the equilibrium contract price is $p_c = \mu_\psi$, and we prove this conjecture in Appendix A.7 below.

Claim 5. If:

$$1+\frac{\alpha\sigma_\eta^2-\kappa}{\alpha\sigma_\psi^2\left(\alpha\kappa\sigma_\eta^2+d^2+2\kappa d\right)}>0 \tag{63}$$

then spot traders' objective function is strictly concave in c_i, and there is a unique optimal choice of c_i, which satisfies:

$$c_i\left(1+\frac{\alpha\sigma_\eta^2-\kappa}{\left(\alpha\sigma_\psi^2\right)\left(d^2+2d\kappa+\alpha\kappa\sigma_\eta^2\right)}\right)=-x_i \tag{64}$$

Proof. Take conditional expected utility from (57). Since only the exponent depends on c, and the function $-\exp\left(-\frac{\alpha}{2}\left(x\right)\right)$ is increasing in x, we choose c_i to maximize:

$$-\alpha\sigma_\psi^2\left(c_i+x_i\right)^2-\frac{\alpha\sigma_\eta^2-\kappa}{\alpha\kappa\sigma_\eta^2+d^2+2\kappa d}c_i^2 \tag{65}$$

Taking the second derivative, note that the problem is only concave if:

$$1+\frac{\alpha\sigma_\eta^2-\kappa}{\alpha\sigma_\psi^2\left(d^2+2d\kappa+\alpha\kappa\sigma_\eta^2\right)}>0$$

proving (63). Assuming (63) holds, differentiate (65) with respect to c_i and rearrange to get (64). Claim 5, plugging in $d=(n-2)\,\kappa$ from (56), gives expressions (9) and (10) of proposition 1. $\square$

A.4 Equilibrium σ_η^2

From (9) of proposition 1, traders' optimal contract purchases are linear in their factor exposures x_i, and traders' factor exposures x_i are mean-0 normal, so traders' equilibrium contract positions are also mean-0 normally distributed. Thus, the residual supply intercept term, η_i, is also mean-0 normally distributed. We can then solve the model by requiring agents' optimal behavior given σ_η^2, the variance of the residual supply intercept η_i, to generate residual supply curves with variance σ_η^2.

Claim 6. For any $\alpha, \sigma_\psi^2, \kappa, \sigma_x^2, n$ there is a unique symmetric equilibrium value of σ_η^2, satisfying:

$$\sigma_\eta^2 = \frac{\sigma_x^2}{n-1}\left(1+\frac{\alpha\sigma_\eta^2-\kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left(\left(n^2-2n\right)\kappa+\alpha\sigma_\eta^2\right)}\right)^{-2} \tag{66}$$

$$\sigma_\eta^2 > \frac{\kappa-\left(\alpha\sigma_\psi^2\right)\left(n^2-2n\right)\kappa^2}{\alpha\left(1+\alpha\sigma_\psi^2\kappa\right)} \tag{67}$$

Proof. Since (64) implies that contract positions c_i are linear in exposures x_i, agents' contract positions are also normally distributed, with mean 0 and variance:

$$\sigma_c^2 = \left(1+\frac{\alpha\sigma_\eta^2-\kappa}{\left(\alpha\sigma_\psi^2\right)\left(d^2+2d\kappa+\alpha\kappa\sigma_\eta^2\right)}\right)^{-2}\sigma_x^2 \tag{68}$$

Now, note that residual supply facing i, summing over other agents' bids using (7), is:

$$-\sum_{j\neq i} z_{Bi}\left(p;c_i\right) = -\sum_{j\neq i}\left(\frac{1}{n-1}c_i-\frac{n-2}{n-1}\kappa\left(p-\psi\right)\right)$$

hence, using representation (50) of residual supply, we have:

$$\eta_i = -\sum_{j\neq i}\frac{1}{n-1}c_i$$

and all factor exposures x_i are independent. Hence, the variance of the residual supply intercept, η_i, is:

$$\sigma_\eta^2 = \frac{\sigma_c^2}{n-1} \tag{69}$$

Combining this with (68), and plugging in $d = (n-2)\,\kappa$ from (56), we obtain (66). Note also that, using the definition of t in (9), we can write (69) as:

$$\sigma_\eta^2 = \frac{t^2 \sigma_x^2}{n-1} \tag{70}$$

By claim 5, in order for (11) to solve agents' optimal contract purchasing problem, the concavity condition (63) must also hold; plugging in $d = (n-2)\,\kappa$ to (63), we get:

$$1 + \frac{\alpha\sigma_\eta^2 - \kappa}{\left(\alpha\sigma_\psi^2 \kappa\right)\left(\left(n^2 - 2n\right)\kappa + \alpha\sigma_\eta^2\right)} > 0 \tag{71}$$

Setting (71) to 0 and solving for σ_η^2, we get:

$$\frac{\kappa - \left(\alpha\sigma_\psi^2\right)\left(n^2 - 2n\right)\kappa^2}{\alpha\left(1 + \alpha\sigma_\psi^2 \kappa\right)} \tag{72}$$

The LHS of (71) is increasing in σ_η^2, so (72) defines a lower bound for equilibrium values of σ_η^2. This is (67) of Claim 6. For notational convenience, we define the lower bound as:

$$M_{lower} \equiv \frac{\kappa - \left(\alpha\sigma_\psi^2\right)\left(n^2 - 2n\right)\kappa^2}{\alpha\left(1 + \alpha\sigma_\psi^2 \kappa\right)} \tag{73}$$

To show that there is a unique value of σ_η^2 satisfying (66) and (67), rearrange (66) to:

$$(n-1)\,\sigma_\eta^2 = \left(1 + \frac{\alpha\sigma_\eta^2 - \kappa}{\left(\alpha\sigma_\psi^2 \kappa\right)\left(\left(n^2 - 2n\right)\kappa + \alpha\sigma_\eta^2\right)}\right)^{-2} \sigma_x^2 \tag{74}$$

We want to study the behavior of (74), for σ_η^2 on the interval (M_{lower}, ∞). This is true because of monotonicity. The LHS of (74) is strictly increasing in σ_η^2, starting from 0 when $\sigma_\eta^2 = 0$, and increasing unboundedly as σ_η^2 is large. On the interval (M_{lower}, ∞), the RHS,

$$\left(1 + \frac{\alpha\sigma_\eta^2 - \kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left(\left(n^2 - 2n\right)\kappa + \alpha\sigma_\eta^2\right)}\right)^{-2} \sigma_x^2$$

is strictly decreasing. As σ_η^2 approaches M_{lower} from the right, the RHS is unbounded above. As σ_η^2 increases towards ∞, the RHS decreases towards the finite quantity:

$$\left(1 + \frac{1}{\alpha\kappa\sigma_\psi^2}\right)^{-2} \sigma_x^2$$

Hence, the LHS and RHS of (74) cross exactly once on the interval (M_{lower}, ∞).

To visually depict this argument, appendix figure A.1 shows the LHS and RHS of (74) under different parameter settings. There are essentially two possibilities: the lower bound M_{lower}, defined in (72), can be positive or negative. The left plot shows a case where M_{lower} is negative. For positive σ_η^2, the RHS begins from a constant value and decreases. The right plot shows a case where M_{lower} is positive. The RHS is ∞ at $\sigma_\eta^2 = M_{lower}$, and decreases towards a constant value. In either case, these curves must cross exactly once, and there is a unique equilibrium value of σ_η^2. □

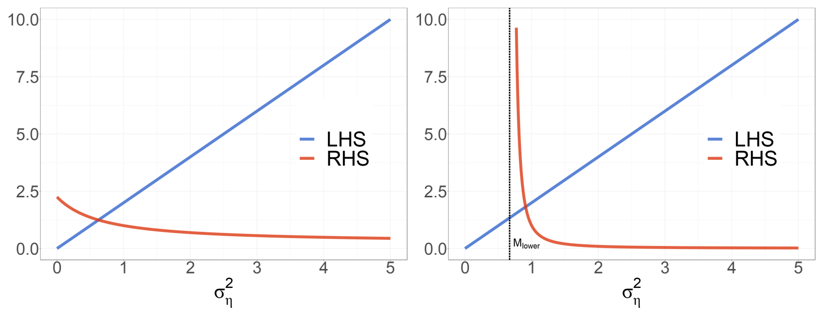
This proves claim 6, and thus (11) and (12) of proposition 1.

A.5 Spot traders' expected welfare over uncertainty in factor exposures

Plugging in $c_i = tx_i$, $d = (n-2)\kappa$ to (57) of claim 4, we get expected utility conditional on x_i, for any linear contract purchasing rule:

$$\sqrt{\frac{\left(n^2 - 2n\right)\kappa}{\alpha\sigma_\eta^2 + \left(n^2 - 2n\right)\kappa}}\left(-\exp\left(-\frac{\alpha}{2}\left(-\alpha\sigma_\psi^2\left(1-t\right)^2 - \frac{\alpha\sigma_\eta^2 - \kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left(\left(n^2 - 2n\right)\kappa + \alpha\sigma_\eta^2\right)}t^2\right)x_i^2\right)\right)$$

Figure A.1: Uniqueness of equilibrium σ_η^2



Notes. Each panel shows the LHS (blue line) and RHS (red line) of (74), as we vary σ_η^2. In the left plot, we set $n = 3, \alpha = 1, \kappa = 1, \sigma_\psi^2 = 1, \sigma_x^2 = 1$. In the right plot, we set $n = 3, \alpha = 0.3, \kappa = 0.3, \sigma_\psi^2 = 1, \sigma_x^2 = 1$. The dotted vertical line in the right plot is the lower bound M_{lower}, defined in (73).

Integrating this against uncertainty in x_i, with mean 0 and variance σ_x^2, we get:

$$-\frac{\sqrt{\frac{(n^2-2n)\kappa}{\alpha\sigma_\eta^2+(n^2-2n)\kappa}}}{\sqrt{1-\alpha\sigma_x^2\left(\alpha\sigma_\psi^2\,(1-t)^2+\left(\frac{\alpha\sigma_\eta^2-\kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left((n^2-2n)\kappa+\alpha\sigma_\eta^2\right)}\right)t^2\right)}} \tag{75}$$

Substituting for σ_η^2 using (70), we get (13).

A.6 The hedger's contract purchasing decisions and welfare

From (4), the hedger's wealth is:

$$W_{hedger}\,(x_H, c_H, p, \psi) = \psi x_H - \mu_\psi x_H - \mu_\psi c_H + p c_H$$

Since the expectation of p, over uncertainty in ψ and x_H, is μ_ψ, the expectation of the hedger's wealth is always equal to 0. To find the variance, adding and subtracting ψc_i, we can write this as:

$$= \psi x_H - \mu_\psi x_H - \mu_\psi c_H + \psi c_H + (p - \psi)\, c_H \tag{76}$$

Since the mean of the hedger's wealth is independent of c_H, the hedger simply purchases contracts to minimize the variance of wealth. Ignoring the constant terms, we can write (76) as:

$$\psi x_H + (\psi + (p - \psi))\, c_H \tag{77}$$

The hedger chooses c_H to minimize the variance of (77), where $p - \psi$ is independent of ψ. This is a regression problem, and the solution is to choose c_H equal to negative the coefficient from regressing ψx_H on $p = \psi + (p - \psi)$:

$$c_H = -\frac{Cov\left(\psi x_i, \psi + (p - \psi)\right)}{Var\left(\psi + (p - \psi)\right)} = -\frac{\sigma_\psi^2 x_H}{\sigma_\psi^2 + Var\left(p - \psi\right)}$$

This proves (14). The residual wealth variance that the hedger is exposed to is $\left(1 - R^2\right)$ times the unhedged variance of wealth, $x_H^2 \sigma_\psi^2$:

$$\begin{aligned} Var\left(\psi x_i + (\psi + (p - \psi))\, c_H\right) &= \left(1 - \frac{\left(Cov\left(x_i \psi, \psi + (p - \psi)\right)\right)^2}{Var\left(\psi + (p - \psi)\right) Var\left(x_i \psi\right)}\right) x_H^2 \sigma_\psi^2 \\ &= \frac{Var\left(p - \psi\right)}{\sigma_\psi^2 + Var\left(p - \psi\right)} x_H^2 \sigma_\psi^2 \end{aligned} \tag{78}$$

Taking the expectation of (78) over x_H, which has variance σ_H^2, we get (15).

A.7 Equilibrium in the contract market

Finally, we show that it is an equilibrium for the market maker to set $p_c = \mu_\psi$. As in Kyle (1985), the market maker sets prices equal to the expectation of prices conditional on order flow. In any linear equilibrium, spot traders' and the pure hedger's contract positions c_i, c_H are linear in their factor exposures x_i, x_H; since we have assumed spot traders' factor exposures are infinitesimal relative to the hedger's, spot traders' contract positions are also infinitesimal. Thus, total order flow for contracts is equal to the hedger's order flow c_H, and the market maker sets price:

$$p_c = E[p \mid c_H]$$

We can write:

$$E[p \mid c_H] = E[\psi + (p - \psi) \mid c_H] = E[\psi \mid c_H] + E[p - \psi \mid c_H]$$

Now, we have assumed c_H is independent of ψ, so $E[\psi \mid c_H] = \mu_\psi$. Moreover, since x_i and x_H are independent, c_i and c_H are independent, so from expression (8) for spot good prices, c_H is independent of $p - \psi$. Hence, $E[p - \psi \mid c_H] = 0$. This shows that

$$E[p \mid c_H] = \mu_\psi$$

Thus, the market maker sets $p_c = \mu_\psi$ in the first stage. This completes the proof of proposition 1.

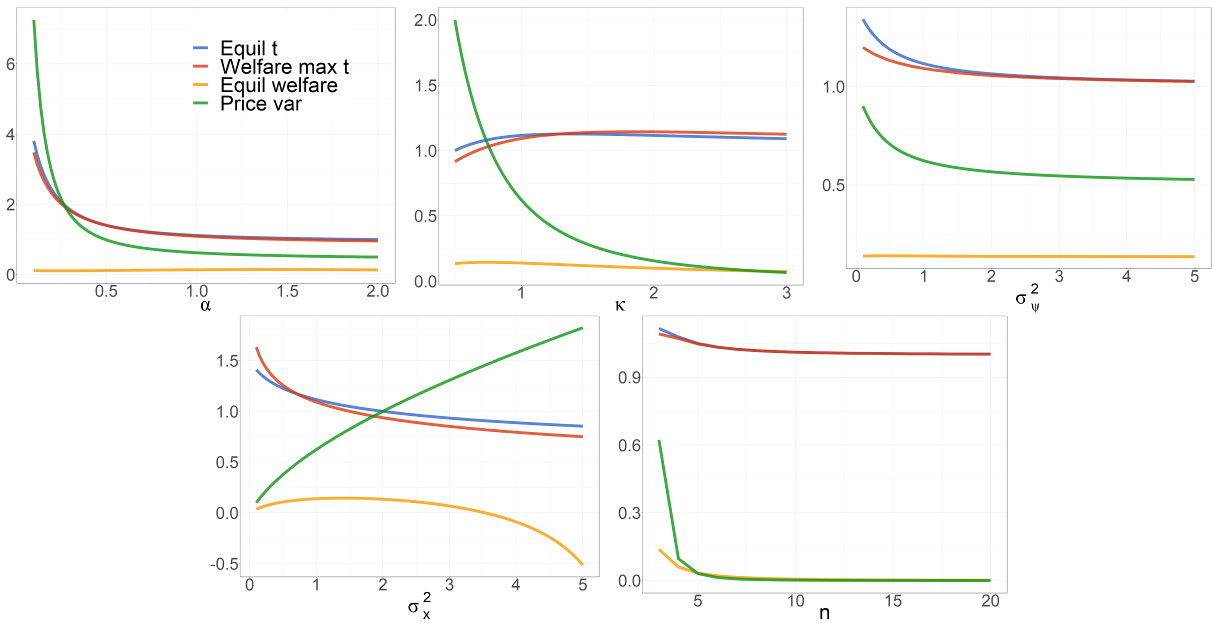
B Supplementary material for section 4

B.1 Comparative statics

Figure A.2 illustrates the effects of varying input parameters on equilibrium outcomes. When spot traders' risk aversion α is low, traders are more willing to bear factor risk in order to attain manipulation profits, so the equilibrium and spot-trader-optimal values of t increase, causing price variance to increase. When the spot good holding capacity κ decreases, price variance increases. The welfare-maximizing value of t for spot traders tends to be lower than the equilibrium t when κ is low, because the negative basis risk externalities from manipulation are larger.

Decreasing the variance of the risk factor, σ_ψ^2, makes spot traders more willing to buy large contract positions and manipulate, which increases t in equilibrium and increases basis risk. Increasing agents' factor exposure variance, σ_x^2, increases price variance, but

Figure A.2: Comparative statics



Notes. Comparative statics of the equilibrium and spot trader welfare-maximizing values of t, spot traders' equilibrium welfare gain (minus the competitive equilibrium value of -1), and equilibrium price variance, $Var(p - \psi)$, as input parameters vary. The baseline values that parameters are varied around are $n = 3$, $\alpha = 1$, $\kappa = 0.8$, $\sigma_\psi^2 = 1.5$, $\sigma_x^2 = 0.05$.

actually decreases the equilibrium and socially optimal values of t, as it becomes more costly for agents to deviate from full hedging. Larger factor exposures also imply that spot traders' welfare losses are larger, and because negative basis risk externalities are larger, the spot-trader welfare maximizing t tends to fall below the equilibrium t when σ_x^2 is large. Finally, increasing n causes all parameters to converge rapidly to their competitive values: the equilibrium t converges to 1, and price variance and net spot trader welfare losses from manipulation converge to 0.

B.2 Unboundedness of equilibrium t

Claim 7. The unique equilibrium value of t is unbounded above: there exist choices of parameters $\alpha, \sigma_\psi^2, \kappa, \sigma_x^2, n$ such that the equilibrium value of t is arbitrarily large.

Proof. From (11) of proposition 1, the equilibrium σ_η^2 must satisfy:

$$(n-1)\,\sigma_\eta^2 = \left(1 + \frac{\alpha\sigma_\eta^2 - \kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left((n^2-2n)\,\kappa + \alpha\sigma_\eta^2\right)}\right)^{-2} \sigma_x^2 \tag{79}$$

We can rearrange this to:

$$\frac{(n-1)\,\sigma_\eta^2}{\sigma_x^2} = \left(1 + \frac{\alpha\sigma_\eta^2 - \kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left((n^2-2n)\,\kappa + \alpha\sigma_\eta^2\right)}\right)^{-2} \tag{80}$$

Analogous to (79), the LHS of (80) is increasing in σ_η^2 and the RHS is decreasing, so the curves cross exactly once. From (10), the RHS of (80) is equal to t^2. Hence, to show that the equilibrium amount of overhedging can be unbounded above, we must show that, in equilibrium, the y-value of the point where the LHS and RHS of (80) cross is unbounded above.

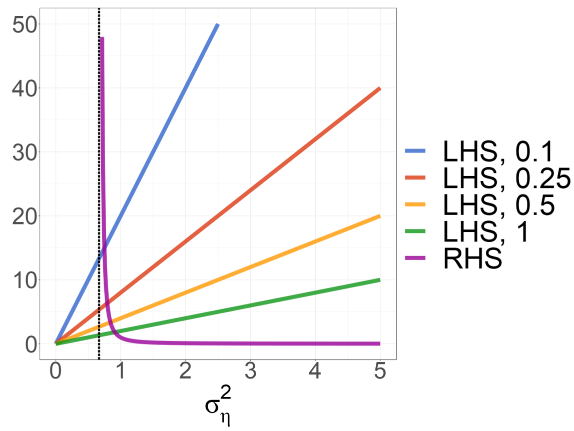
To show this, consider any set of parameters such that the lower bound M_{lower}, defined in (73), is greater than 0. This is true when:

$$\left(\alpha\sigma_\psi^2\right)\left(n^2-2n\right)\kappa < 1$$

Intuitively, this requires that $n, \alpha, \sigma_\psi^2, \kappa$ are relatively small. The RHS of (80) then approaches ∞ as σ_η^2 approaches M_{lower} from the right. As we decrease σ_x^2 towards 0, the y-value of the intercept increases unboundedly. This is depicted in Appendix Figure A.3 below. As we decrease σ_x^2, the LHS of (80) becomes steeper, and intersects the RHS (the purple line) at higher and higher y-values. The RHS is unbounded above, so the y-value of the intercept is also unbounded above.

Intuitively, what is happening in this example is that the positive lower bound M_{lower}

Figure A.3: Unbounded over-hedging



Notes. The dotted vertical line is the lower bound M_{lower}, defined in (73). The purple line is the RHS of (80), as σ_η^2 varies. The blue, red, yellow, and green lines show the LHS of (80), for different values of σ_x^2 in $\{0.1, 0.25, 0.5, 1\}$. For all lines, we set $n = 3, \alpha = 0.3, \kappa = 0.3, \sigma_\psi^2 = 1$.

implies that there cannot be an equilibrium with low σ_η^2, in order for spot traders' second-order condition to hold. But σ_η^2 depends on t and σ_x^2 , the variance of spot traders' factor risk exposures. As we decrease σ_x^2 towards 0, in order to sustain an equilibrium with a large equilibrium σ_η^2, spot traders must buy a large number of contracts for each unit of their factor risk exposures, so t must increase unboundedly to keep σ_η^2 above the lower bound M_{lower}. In other words, spot traders must manipulate enough, and create enough basis risk, to discourage other spot traders from buying infinitely large contract positions. $\square$

B.3 Convergence rate of equilibrium t

Claim 8. As n increases to ∞ holding fixed other parameters, $t - 1$ converges to 0 at rate $\frac{1}{n^2}$.

Proof. From (11) and (10), we have:

$$\sigma_\eta^2 = \frac{\sigma_x^2}{n-1}\left(1+\frac{\alpha\sigma_\eta^2-\kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left(\left(n^2-2n\right)\kappa+\alpha\sigma_\eta^2\right)}\right)^{-2} \tag{81}$$

and:

$$t = \left(1+\frac{\alpha\sigma_\eta^2-\kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left(\left(n^2-2n\right)\kappa+\alpha\sigma_\eta^2\right)}\right)^{-2} \tag{82}$$

The RHS of (82) is decreasing in σ_η^2. Hence an upper bound for the equilibrium t comes from setting σ_η^2 to 0:

$$t \leqslant \left(1-\frac{\kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left(\left(n^2-2n\right)\kappa\right)}\right)^{-1} \tag{83}$$

Now, we can get an upper bound for σ_η^2 by plugging the upper bound on t, (83) into (81):

$$\sigma_\eta^2 \leqslant \frac{\sigma_x^2}{n-1}\left(1-\frac{\kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left(\left(n^2-2n\right)\kappa\right)}\right)^{-2} \tag{84}$$

The RHS is decreasing in n, hence, we can set $n=3$ in (84) to get:

$$\sigma_\eta^2 \leqslant M \equiv \frac{\sigma_x^2}{2}\left(1-\frac{\kappa}{\left(\alpha\sigma_\psi^2\kappa\right)(3\kappa)}\right)^{-2}$$

where M does not depend on n. Now, a lower bound for t comes from plugging the upper bound M into (82).

$$t \geqslant \left(1-\frac{\alpha M-\kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left(\left(n^2-2n\right)\kappa+\alpha M\right)}\right)^{-1} \tag{85}$$

Together, (83) and (85) bound the equilibrium value of t. Now, the function $\frac{1}{1-x}$ is differentiable at $x=1$ with derivative equal to 1. Hence, (83) converges to 1 at the same rate that

$$1-\frac{\kappa}{\left(\alpha\sigma_{\psi}^{2}\kappa\right)\left(\left(n^{2}-2n\right)\kappa\right)}$$

converges to 1, which is $\frac{1}{n^2}$; (85) is analogous. Thus, the equilibrium value of $t-1$ converges to 0 at rate $\frac{1}{n^2}$. □

B.4 Heterogeneous beliefs

In this appendix, I show that contract purchases can also be generated by dispersion in agents' beliefs about ψ. Suppose a spot trader believes the mean of ψ is $\beta_{\psi i}$, and its variance is σ_{ψ}^{2}. We assume for simplicity, as in proposition 1, that spot traders can purchase contracts at a fixed price μ_{ψ}, and have no price impact in the contract market. We assume spot traders have no factor exposures. From (62), the agent's utility as a function of ψ, over uncertainty in the spot market, is:

$$\sqrt{\frac{d^{2}+2\kappa d}{\alpha\kappa\sigma_{\eta}^{2}+d^{2}+2\kappa d}}\left(-\exp\left(-\alpha\left(\psi c_{i}-\mu_{\psi}c_{i}\right)-\frac{\alpha}{2}\left(\frac{\alpha\sigma_{\eta}^{2}-\kappa}{\alpha\kappa\sigma_{\eta}^{2}+d^{2}+2\kappa d}\right)c_{i}^{2}\right)\right)$$

To calculate the agent's expected utility (under her beliefs), we integrate assuming ψ has mean $\beta_{\psi i}$ and variance σ_{ψ}^{2}. This gives:

$$=\sqrt{\frac{d^{2}+2\kappa d}{\alpha\kappa\sigma_{\eta}^{2}+d^{2}+2\kappa d}}\left(-\exp\left(-\frac{\alpha}{2}\left(-\alpha\sigma_{\psi}^{2}c_{i}^{2}+2\beta_{\psi i}c_{i}-2\mu_{\psi}c_{i}-\frac{\alpha\sigma_{\eta}^{2}-\kappa}{\alpha\kappa\sigma_{\eta}^{2}+d^{2}+2\kappa d}c_{i}^{2}\right)\right)\right)$$

Only the exponent depends on c_i, so we maximize:

$$-\alpha\sigma_{\psi}^{2}c_{i}^{2}+2\left(\beta_{\psi i}-\mu_{\psi}\right)c_{i}-\frac{\alpha\sigma_{\eta}^{2}-\kappa}{\alpha\kappa\sigma_{\eta}^{2}+d^{2}+2\kappa d}c_{i}^{2}$$

Differentiating and solving for c_i, we have:

$$c_{i}\left(1+\frac{\alpha\sigma_{\eta}^{2}-\kappa}{\left(\alpha\sigma_{\psi}^{2}\right)\left(d^{2}+2d\kappa+\alpha\kappa\sigma_{\eta}^{2}\right)}\right)=\frac{\beta_{\psi i}-\mu_{\psi}}{\alpha\sigma_{\psi}^{2}} \tag{86}$$

Comparing (86) to (9) and (10), a belief shock $\beta_{\psi i}$ is isomorphic to a factor exposure x_i of size:

$$x_i = -\frac{\beta_{\psi i} - \mu_\psi}{\alpha \sigma_\psi^2}$$

in the sense that it generates the same contract purchasing decisions and spot market behavior. Hence, contract purchases can be motivated either by disagreement or risk-sharing. If we do not observe factor exposures directly, based only on agents' contract purchases and spot market behavior, we cannot separately identify factor exposures from heterogeneous beliefs, complicating welfare analysis for spot traders.

However, spot traders' optimal behavior in spot markets only depends on the size of their contract positions and the structure of spot markets, not the motivations of spot traders for purchasing contracts. Hence, the results of this paper concerning spot traders' incentives in the spot market, and their effects on hedgers' welfare, hold even if spot traders' contract positions are driven by differences in beliefs.

B.5 Price impact in the contract market, quadratic taxes and subsidies

In this appendix, I show that, if spot traders' contract purchases have some exogeneous amount of price impact, we can still solve for linear optimal contract purchasing rules. Moreover, price impact is isomorphic to subsidies or taxes to spot traders which are quadratic in the size of traders' contract positions. Thus, a regulator can implement any desired choice of hedging aggressiveness t in equilibrium using some quadratic subsidy or tax scheme.

Suppose that spot traders' contract purchases move the spot price linearly: if a spot trader purchases c_i contracts, the price per contract is:

$$\mu_\psi + \lambda c_i$$

The total cost of buying c_i contracts is then:

$$\mu_\psi c_i + \lambda c_i^2 \tag{87}$$

Combining this cost with (3) and integrating, a spot trader's conditional expected utility if she has factor exposure x_i and contract position c_i is:

$$\frac{-1}{\sqrt{2\pi\sigma_\psi^2}}\int\sqrt{\frac{d^2+2\kappa d}{\alpha\kappa\sigma_\eta^2+d^2+2\kappa d}}\exp\Bigg[$$

$$-\alpha\left(\psi x_i-\mu_\psi x_i+\psi c_i-\mu_\psi c_i-\lambda c_i^2\right)-\frac{\alpha}{2}\left(\frac{\alpha\sigma_\eta^2-\kappa}{\alpha\kappa\sigma_\eta^2+d^2+2\kappa d}\right)c_i^2\Bigg]\exp\left(-\frac{\left(\psi-\mu_\psi\right)^2}{2\sigma_\psi^2}\right)d\psi$$

$$=\sqrt{\frac{d^2+2\kappa d}{\alpha\kappa\sigma_\eta^2+d^2+2\kappa d}}\left(-\exp\left(-\frac{\alpha}{2}\left(-\alpha\sigma_\psi^2\left(c_i+x_i\right)^2-2\lambda c_i^2-\frac{\alpha\sigma_\eta^2-\kappa}{\alpha\kappa\sigma_\eta^2+d^2+2\kappa d}c_i^2\right)\right)\right)$$

Hence, agents choose c_i to maximize:

$$-\alpha\sigma_\psi^2\left(c_i+x_i\right)^2-2\lambda c_i^2-\frac{\alpha\sigma_\eta^2-\kappa}{\alpha\kappa\sigma_\eta^2+d^2+2\kappa d}c_i^2$$

The optimal choice of c_i thus satisfies:

$$c_i=-\left(1+\frac{\alpha\sigma_\eta^2-\kappa}{\left(\alpha\sigma_\psi^2\right)\left(d^2+2d\kappa+\alpha\kappa\sigma_\eta^2\right)}+\frac{4\lambda}{\alpha\sigma_\psi^2}\right)^{-1}x_i \tag{88}$$

Substituting for d using the spot market equilibrium bid curves, we have:

$$c_i=-\left(1+\frac{\alpha\sigma_\eta^2-\kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left(\left(n^2-2n\right)\kappa+\alpha\sigma_\eta^2\right)}+\frac{4\lambda}{\alpha\sigma_\psi^2}\right)^{-1}x_i \tag{89}$$

Expression (89) shows that spot traders' optimal contract positions are still linear in their factor exposures x_i, with coefficient:

$$t\equiv\left(1+\frac{\alpha\sigma_\eta^2-\kappa}{\left(\alpha\sigma_\psi^2\kappa\right)\left(\left(n^2-2n\right)\kappa+\alpha\sigma_\eta^2\right)}+\frac{4\lambda}{\alpha\sigma_\psi^2}\right)^{-1} \tag{90}$$

Comparing (90) to (10) of proposition 1, price impact in the contract market decreases

t, causing spot traders to hedge less per unit of their factor exposures. Given this t, the equilibrium σ_η^2 is the unique value that satisfies:

$$\sigma_\eta^2 = \frac{\sigma_x^2}{n-1}\left(1 + \frac{\alpha\sigma_\eta^2 - \kappa}{\left(\alpha\sigma_\psi^2 \kappa\right)\left(\left(n^2 - 2n\right)\kappa + \alpha\sigma_\eta^2\right)} + \frac{4\lambda}{\alpha\sigma_\psi^2}\right)^{-2}$$

and outcomes in the spot market are described by (7) and (8). Qualitatively, price impact in contract markets simply causes spot traders to buy less contracts per unit of their factor exposures.

Expressions (88) and (90) also imply that regulators can implement any desired choice of t as a unique equilibrium, by imposing quadratic taxes or subsidies on agents' contract positions. To see this, suppose that a regulator can charge all spot traders some net amount τc_i^2 for buying c_i contracts, where τ can be positive or negative. A spot trader's total cost for buying c_i contracts is thus:

$$\mu_\psi c_i + \lambda c_i^2 + \tau c_i^2$$

Analogous to (90), spot traders' optimal hedging decisions are linear, satisfying:

$$t \equiv \left(1 + \frac{\alpha\sigma_\eta^2 - \kappa}{\left(\alpha\sigma_\psi^2\right)\left(d^2 + 2d\kappa + \alpha\kappa\sigma_\eta^2\right)} + \frac{4\lambda}{\alpha\sigma_\psi^2} + \frac{4\tau}{\alpha\sigma_\psi^2}\right)^{-1} \tag{91}$$

Now, suppose the regulator wishes to implement some positive level of hedging aggressiveness, $t^* > 0$, in equilibrium. From (70) of appendix A.4, we have:

$$\sigma_\eta^2 = \frac{\left(t^*\right)^2 \sigma_x^2}{n-1}$$

Hence, in order to implement t^*, we must choose τ such that:

$$t^* = \left(1 + \frac{\alpha\left(\frac{(t^*)^2\sigma_x^2}{n-1}\right) - \kappa}{\left(\alpha\sigma_\psi^2\right)\left(d^2 + 2d\kappa + \alpha\kappa\left(\frac{(t^*)^2\sigma_x^2}{n-1}\right)\right)} + \frac{4\lambda}{\alpha\sigma_\psi^2} + \frac{4\tau}{\alpha\sigma_\psi^2}\right)^{-1}$$

By changing τ, the RHS can be varied from 0 to ∞, so for any t^*, and any values of other primitives, there is a unique value of τ which implements t^* as an equilibrium outcome.

Throughout this appendix, we have taken price impact as exogenous. In a more realistic model, price impact would result endogenously from the contract market, in which spot traders and the hedger bid for contracts in a double auction. This is analytically difficult to solve, because spot traders' order flow would be informative about their spot market trades, and thus contract settlement prices.

The λc_i^2 term in (87) could also be used to model holding costs for contracts, which may arise from margin capital requirements or related factors. The results of this appendix then imply that quadratic holding costs, like price impact, decrease spot traders' hedging aggressiveness in equilibrium. Once again, spot traders' optimal behavior in spot markets depends only on the size of their contract positions and spot market structure. Thus, price impact in contract markets would not affect the results of section 5.

C Supplementary material for section 5

C.1 Proof of propositions 2 and 3

We prove proposition 3; proposition 2 is a special case. Claim 9 first characterizes agents' best responses, given the slope of residual supply, and subsection C.1.2 proves proposition 3 using claim 9.

C.1.1 Best responses

Claim 9. If agent i has inventory position y_i and contract position c_i, and the slope of residual supply is d_i, then agent i's unique ex-post optimal bid curve is:

$$z_{Bi}(p; y_i, c_i) = -\frac{d_i}{\kappa_i + d_i} y_i + \frac{\kappa_i}{\kappa_i + d_i} c_i - \frac{\kappa_i d_i}{\kappa_i + d_i}(p - \psi) \tag{92}$$

Proof. Analogously to claim 3, assume that residual supply takes the form:

$$z_{RSi}(p, \eta) = d_i (p - \psi) + \eta_i$$

We will optimize pointwise in the residual supply intercept η_i. Since we assumed all c_i and y_i have full support, in any linear bidding equilibrium, all η_i's will also have full support. Define $p^*(\eta_i)$ as the optimal choice of p for any given η_i, that is:

$$p^*(\eta_i) \equiv \arg\max_p W(z_{RSi}(p, \eta_i), p; y_i, c_i)$$

$$= \arg\max_p \psi z_{RSi}(p, \eta_i) - \frac{y_i^2}{2\kappa_i} - \frac{y_i z_{RSi}(p, \eta_i)}{\kappa_i} - \frac{z_{RSi}(p, \eta_i)^2}{2\kappa_i} + c_i p - z_{RSi}(p, \eta_i) p$$

Since $z_{RSi}(p, \eta_i)$ is affine and increasing in p, the objective function concave in p, thus the first-order condition is necessary and sufficient for $p^*(\eta_i)$ to be optimal. Differentiating with respect to p and setting to 0, and using that $z'_{RSi}(p, \eta_i) = d_i$, we have:

$$-\frac{d_i}{\kappa_i} y_i - \frac{z_{RS}(p^*(\eta_i), \eta_i)}{\kappa_i} d_i + c_i - z_{RS}(p^*(\eta_i), \eta_i) - (p^*(\eta_i) - \pi) d_i = 0 \tag{93}$$

Hence, any pair $(p^*(\eta_i), z_{RS}(p^*(\eta_i), \eta_i))$ – that is, any point (p, z) which is the agent's optimal choice for some η_i – satisfies (93). Hence, the unique bid curve which passes through the set of all ex-post optimal points is the curve implicitly defined by (93). Solving (93) for $z_{RS}(p^*(\eta_i), \eta_i)$, we have (92). □

C.1.2 Equilibrium

This proof is based on Appendix A.4 of Du and Zhu (2012), with notational modifications to suit the context of this paper. We seek a vector of demand and residual supply slopes b_i which satisfy, for all i:

$$d_i = \sum_{j \neq i} b_i = B - b_i \tag{94}$$

$$b_i = \frac{d_i \kappa_i}{\kappa_i + d_i} \tag{95}$$

Rearranging, we have:

$$d_i = \frac{b_i \kappa_i}{\kappa_i - b_i} \tag{96}$$

Combining (94) and (96), we have:

$$\sum_j b_j - b_i = \frac{b_i \kappa_i}{\kappa_i - b_i}$$

Defining $B \equiv \sum_j b_j$, we have

$$(\kappa_i - b_i)(B - b_i) = b_i \kappa_i$$

This has two solutions. In order for $B > b_i$, we must pick:

$$b_i = \frac{2\kappa_i + B - \sqrt{B^2 + 4\kappa_i^2}}{2} \tag{97}$$

This is (22) of proposition 2. B must satisfy:

$$B = \sum_j b_j = \sum_{i=1}^{n} \frac{2\kappa_i + B - \sqrt{B^2 + 4\kappa_i^2}}{2} \tag{98}$$

This is (23) in the main text. By multiplying the top and bottom of the RHS by $2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}$ and simplifying, this becomes:

$$B = \sum_{i=1}^{n} \frac{2\kappa_i B}{2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}}$$

Or,

$$B \left(-1 + \sum_{i=1}^{n} \frac{2\kappa_i}{2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}} \right) = 0 \tag{99}$$

Now, define

$$f(B) = -1 + \sum_{i=1}^{n} \frac{2\kappa_i}{2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}}$$

In order for B to solve (99) when $B > 0$, we need $f(B) = 0$. Now, $f(0) > 0$, $f(B) \to -1$ as $B \to \infty$, and $f'(B) < 0$ for $B > 0$. Hence, $f(B) = 0$ at some unique B, hence there is a unique value of B which solves (99), and thus there is a unique linear equilibrium for any demand slopes $\kappa_1 \dots \kappa_n$.

Substituting $b_i = \frac{\kappa_i d_i}{\kappa_i + d_i}$ into agents' best-response bid curves from claim 9, we have agents' equilibrium bids in expression (30). To find prices, sum agents' demand curves and equate to 0:

$$\sum_{i=1}^{n} \left[-y_i \frac{b_i}{\kappa_i} + c_i \frac{b_i}{\sum_{j \neq i} b_j} - (p - \psi) b_i \right] = 0$$

Solving for p, we have (31).

C.2 Proof of claim 2

Proof. If contract positions are i.i.d., with variance σ_c^2, the variance of prices can be calculated by taking the variance of (21):

$$Var(p - \psi) = \frac{1}{B^2} \sum_{i=1}^{n} \left(\frac{b_i}{\sum_{j \neq i} b_j} \right)^2 \sigma_c^2 \tag{100}$$

Now, using the upper and lower bounds we have upper and lower bounds for each $\frac{b_i}{\sum_{j \neq i} b_j}$ from (25) of claim 1, we have:

$$\frac{1}{B^2} \sum_{i=1}^{n} s_i^2 \sigma_c^2 \leqslant \frac{1}{B^2} \sum_{i=1}^{n} \left(\frac{b_i}{\sum_{j \neq i} b_j} \right)^2 \sigma_c^2 \leqslant \frac{1}{B^2} \sum_{i=1}^{n} \left(1 + \frac{s_{max}}{1 - 2s_{max}} \right)^2 s_i^2 \sigma_c^2$$

$$\implies \frac{\sigma_c^2}{B^2} \sum_{i=1}^{n} s_i^2 \leqslant \frac{1}{B^2} \sum_{i=1}^{n} \left(\frac{b_i}{\sum_{j \neq i} b_j} \right)^2 \sigma_c^2 \leqslant \frac{\sigma_c^2}{B^2} \left(1 + \frac{s_{max}}{1 - 2s_{max}} \right)^2 \sum_{i=1}^{n} s_i^2$$

Using the definition of HHI in (26), we get (27). □

C.3 Proof of corollary 1

Conjecture that:

$$b_i = \frac{n-2}{n-1}\kappa$$

This solves (22) and (23). Plugging this into (30) and (31), we get (34). To calculate price variance, set the sum of all agents' bids to 0 and solve for price, to get:

$$p - \psi = \frac{1}{n\kappa}\sum_{i=1}^{n}\left[-y_i + \frac{1}{n-2}c_i\right]$$

Taking the variance, and using that all y_i and c_i are i.i.d., we get (35).

C.4 Proof of proposition 4

For basis risk, we simply take the variance of expression (31). Let $\mathbf{y}$ and $\mathbf{c}$ represent vectors of inventory shocks and contract positions:

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Writing (31) using vector notation, prices are:

$$p - \psi = \frac{1}{B}\left(\mathbf{k}_y'\mathbf{y} + \mathbf{k}_c'\mathbf{c}\right)$$

Since we have assumed that contract positions and inventory shocks have mean 0, price variance can be written as:

$$\mathsf{E}\left[(p-\psi)^2\right] = \mathsf{E}\left[\frac{1}{B^2}\left(\mathbf{k}_y'\mathbf{y} + \mathbf{k}_c'\mathbf{c}\right)\left(\mathbf{k}_y'\mathbf{y} + \mathbf{k}_c'\mathbf{c}\right)'\right] = \mathsf{E}\left[\frac{1}{B^2}\left(\mathbf{k}_y'\mathbf{y}\mathbf{y}'\mathbf{k}_y + 2\mathbf{k}_y'\mathbf{y}\mathbf{c}'\mathbf{k}_c + \mathbf{k}_c'\mathbf{c}\mathbf{c}'\mathbf{k}_c\right)\right] \tag{101}$$

Now, since we assumed each element of $\mathbf{y}$ and $\mathbf{c}$ has mean 0, we have:

$$\mathsf{E}\left[\mathbf{y}\mathbf{y}'\right] = \mathbf{\Sigma}_{\mathbf{yy}},\ \mathsf{E}\left[\mathbf{y}\mathbf{c}'\right] = \mathbf{\Sigma}_{\mathbf{yc}},\ \mathsf{E}\left[\mathbf{c}\mathbf{c}'\right] = \mathbf{\Sigma}_{\mathbf{cc}} \tag{102}$$

Substituting these into (101), we get (37).

C.5 Manipulation rents

Define the expected manipulation rent as:

$$\mathrm{E}\left[\left(p-\mu_{\psi}\right)\sum_{i=1}^{n}c_i\right] \tag{103}$$

Expression (104) is the total expected transfer from spot traders to manipulators. I call this the *expected manipulation rent*, as it represents the rents that spot traders, as a group, extract from the hedger, as a result of spot traders' ability to move prices in spot markets. This may be a useful number in regulatory proceedings, to quantify the total expected profits that spot traders as a group extract from the representative hedger. In the symmetric case, the expected manipulation rent is:

Claim 10. In the symmetric model, the expected manipulation rent is:

$$\tau \equiv \mathrm{E}\left[\left(p-\mu_{\psi}\right)\sum_{i=1}^{n}c_i\right] = \frac{\sigma_c^2}{(n-2)\,\kappa} \tag{104}$$

Proof. Since we have assumed that ψ is independent of factor exposures x_i, which are linearly related to contract positions c_i, we have:

$$\mathrm{E}\left[\psi\sum_{i=1}^{n}c_i\right] = \mathrm{E}\left[\psi\right]\mathrm{E}\left[\sum_{i=1}^{n}c_i\right] = \mathrm{E}\left[\mu_{\psi}\sum_{i=1}^{n}c_i\right]$$

Hence,

$$\mathrm{E}\left[\left(p-\mu_{\psi}\right)\sum_{i=1}^{n}c_i\right] = \mathrm{E}\left[\left(p-\psi\right)\sum_{i=1}^{n}c_i\right]$$

Now, plugging in for $(p-\psi)$ using (8) of proposition 1, this is:

$$\mathrm{E}\left(\left(\sum_{i=1}^{n}c_i\right)\frac{\left(\sum_{i=1}^{n}c_i\right)}{n\,(n-2)\,\kappa}\right) \tag{105}$$

Since we have assumed agents' factor exposures x_i are independent, agents' contract positions c_i are also independent, so (105) becomes (104). □

Expression (104) behaves similarly to basis risk, (15). Both are higher when spot traders' contract positions, σ_c^2, are large; when spot traders' holding capacities κ are low, so the price impact of spot trades is higher; and when n is lower, so auctions are less competitive.

In the context of the model, only basis risk matters for the hedger's welfare. This is because we have made the simplifying assumption that spot traders are infinitisemally small, so manipulation rents are infinitesimally small from the perspective of the hedger. Basis risk, on the other hand, affects the hedger and is not diffused away. The following claim shows that manipulation rents are also fairly simple to estimate in the general model, with similar data requirements to basis risk.

Claim 11. In the general model of proposition 3, manipulation rents are:

$$\tau = \frac{1}{B}\left(-2\mathbf{k}_y'\mathbf{\Sigma}_{yc}\mathbf{1} + \mathbf{k}_c'\mathbf{\Sigma}_{cc}\mathbf{1}\right) \tag{106}$$

where $\mathbf{1}$ is the length-n unit vector.

Proof. We have:

$$E\left[(p-\psi)\left(\sum_{i=1}^{n} c\right)\right] = E\left[\left(\frac{1}{B}\left(\mathbf{k}_y'\mathbf{y} + \mathbf{k}_c'\mathbf{c}\right)\right)\left(\mathbf{c}'\mathbf{1}\right)\right] = E\left[\frac{1}{B}\left(\mathbf{k}_y'\mathbf{y}\mathbf{c}'\mathbf{1} + \mathbf{k}_c'\mathbf{c}\mathbf{c}'\mathbf{1}\right)\right]$$

Substituting using (102), we get (106). □

C.6 Proof of proposition 5

In the competitive case, spot traders' wealth is described by (41). Spot traders' marginal value of the spot good is thus:

$$\psi - \frac{c_i}{\kappa} - \frac{z_i}{\kappa}$$

If residual supply is perfectly elastic, spot traders bid to equate marginal values to the spot price:

$$\psi - \frac{c_i}{\kappa} - \frac{z_i}{\kappa} = p$$

Solving for z_i, we get (43).

C.7 Proof of proposition 6

We largely follow appendix A.1. Traders face residual supply curve:

$$z_{RSi}(p) = d(p - \psi) + \eta_i$$

Spot traders' wealth, ignoring terms independent of z_i, is:

$$(z_i + c_i)\psi - \frac{1}{2\kappa}(z_i + c_i)^2 - p_{RS}(z_i; \eta_i, \psi) z_i \tag{107}$$

If we write (107) in terms of the quantity that traders take delivery of, $q_i \equiv z_i - c_i$, defined in (42), we get:

$$\begin{aligned} & q_i \psi - \frac{1}{2\kappa} q_i^2 - p_{RS}(z_i; \eta_i, \psi)(q_i - c_i) \\ = {} & p_{RS}(z_i; \eta_i, \psi) c_i + q_i \psi - \frac{q_i^2}{2\kappa} - p_{RS}(z_i; \eta_i, \psi) q_i \end{aligned} \tag{108}$$

Expression (108) is identical to (53), up to terms that do not depend on c_i, z_i, p_{RS}. Thus, appendix A.1 implies that the ex-post equilibrium bid curves, expressed in terms of delivered quantities q_i, are:

$$q_{Bi}(p; c_i, \psi) = \frac{1}{n-1} c_i - \frac{n-2}{n-1} \kappa (p - \psi) \tag{109}$$

this proves (46) of proposition 6. Substituting z_{Bi} for q_{Bi} in (109) using the fact that $q_{Bi} = z_{Bi} + c_i$ from (42), we get (45) of proposition 6. To calculate equilibrium prices, note that markets clear when:

$$\sum_{i=1}^{n} c_i + z_{Bi}(q; c_i, \psi) = \sum_{i=1}^{n} q_{Bi}(p; c_i, \psi) = 0$$

Since the quantity-delivered bid curve (109) is identical to the bid curve (7) in proposition 1 of the baseline model, the market clearing price under physical delivery is identical to the market clearing price (8) of proposition 1 in the baseline model, proving (47).

D Proof of claim 1

Fix $\kappa_1 \ldots \kappa_N$, and let

$$s_{max} = \frac{\max_i \kappa_i}{\sum_{i=1}^{n} \kappa_i}$$

In appendix D.1 below, I prove the following claim.

Claim 12. For any demand slopes $\kappa_1 \ldots \kappa_N$, when $s_{max} < \frac{1}{2}$, for all i, we have:

$$\left(1 - \frac{s_{max}}{1 - s_{max}}\right) \leqslant \frac{b_i}{\kappa_i} \leqslant 1$$

Now, from claim 9 of appendix C.1, agents' best-response bid curves are:

$$z_{Bi}\left(p; y_i, c_i\right) = \frac{d_i}{\kappa_i + d_i} y_i + \frac{\kappa_i}{\kappa_i + d_i} c_i - \frac{\kappa_i d_i}{\kappa_i + d_i}\left(p - \pi\right) \tag{110}$$

We want to bound the difference between this and the approximation:

$$z_{Bi}\left(p; y_i, c_i\right) \approx y_i + \frac{\kappa_i}{\sum_{i=1}^{n} \kappa_i} c_i - \kappa_i\left(p - \pi\right) \tag{111}$$

Using claim 12, we can bound all three coefficients in the bid curve (110). Recall that $b_i = \frac{\kappa_i d_i}{\kappa_i + d_i}$. Thus, we have:

$$\left(1 - \frac{s_{max}}{1 - s_{max}}\right) \kappa_i \leqslant \frac{\kappa_i d_i}{\kappa_i + d_i} \leqslant \kappa_i$$

$$1 - \frac{s_{max}}{1 - s_{max}} \leqslant \frac{d_i}{\kappa_i + d_i} \leqslant 1$$

Recall that $d_i \equiv \sum_{j \neq i} b_i$. Thus,

$$\frac{\kappa_i}{\kappa_i + d_i} = \frac{\kappa_i}{\kappa_i + \sum_{j \neq i} b_i}$$

Using claim 12, we have:

$$\frac{\kappa_i}{\kappa_i + \sum_{j \neq i} \kappa_j} \geqslant \frac{\kappa_i}{\kappa_i + \sum_{j \neq i} b_i} \geqslant \frac{\kappa_i}{\kappa_i + \left(1 - \frac{s_{max}}{1 - s_{max}}\right) \sum_{j \neq i} \kappa_j} \tag{112}$$

Now, note that:

$$\frac{\kappa_i}{\kappa_i + \left(1 - \frac{s_{max}}{1 - s_{max}}\right) \sum_{j \neq i} \kappa_j} \geqslant \frac{\kappa_i}{\left(1 - \frac{s_{max}}{1 - s_{max}}\right) \left(\kappa_i + \sum_{j \neq i} \kappa_j\right)} = s_i \left(1 + \frac{s_{max}}{1 - 2s_{max}}\right)$$

Hence, (112) becomes:

$$s_i \leqslant \frac{\kappa_i}{\kappa_i + \sum_{j \neq i} b_i} \leqslant \left(1 + \frac{s_{max}}{1 - 2s_{max}}\right) s_i$$

This proves claim 1, since from (110), $\frac{\kappa_i}{\kappa_i + \sum_{j \neq i} b_i}$ is just another way to write $\frac{b_i}{\sum_{j \neq i} b_j}$.

D.1 Proof of claim 12

From (22) in proposition 2, we have:

$$b_i = \frac{B + 2\kappa_i - \sqrt{B^2 + 4\kappa_i^2}}{2}$$

This immediately implies that $b_i \leqslant \kappa_i$, as b_i approaches κ_i from below as $B \to \infty$. This proves the upper bound in claim 12,

$$\frac{b_i}{\kappa_i} \leqslant 1$$

Thus, we need only prove the lower bound:

$$\left(1 - \frac{s_{max}}{1 - s_{max}}\right) \leqslant \frac{b_i}{\kappa_i} \tag{113}$$

We will proceed in two stages. Appendix D.1.1 proves claim 13, which states that, given s_{max} and κ_{max}, we can construct an analytical lower bound for B. Appendix D.1.2 then uses the lower bound for B to lower-bound the ratio $\frac{b_i}{\kappa_i}$.

D.1.1 A lower bound for B

This subsection proves the following claim:

Claim 13. Fixing κ_{max}, s_{max}, we have:

$$B \geqslant \frac{2s_{max} - 1}{s_{max}\left(s_{max} - 1\right)} \kappa_{max}$$

Proof. From (23) in proposition 2, given $\kappa_1 \ldots \kappa_N$, B satisfies:

$$-1 + \sum_{i=1}^{n} \frac{2\kappa_i}{2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}} = 0 \tag{114}$$

For any $\kappa_1 \ldots \kappa_N$, define $B\left(\kappa_1 \ldots \kappa_n\right)$ as the induced value of B, that is:

$$B\left(\kappa_1 \ldots \kappa_n\right) = \left\{ B : \; -1 + \sum_{i=1}^{n} \frac{2\kappa_i}{2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}} = 0 \right\}$$

Now, without loss of generality, suppose agent $i = 1$ has the largest demand slope, so that:

$$s_{max} = s_1 = \frac{\kappa_1}{\sum_{i=1}^{n} \kappa_i}$$

Alternatively, we can write this as:

$$\sum_{i=2}^{n} \kappa_i = \left(\frac{1 - s_{max}}{s_{max}}\right) \kappa_1 \tag{115}$$

Consider the following optimization problem:

$$\begin{aligned} &\min_{\kappa_2 \dots \kappa_n} \mathrm{B}\left(\kappa_1 \dots \kappa_n\right) \\ &\text{s.t. } \sum_{i=2}^{n} \kappa_i = \left(\frac{1 - s_{max}}{s_{max}}\right) \kappa_1,\ 0 \leqslant \kappa_i \leqslant \kappa_1\ \forall i \end{aligned} \tag{116}$$

In words, problem (116) states that we choose $\kappa_2 \dots \kappa_n$ to minimize $\mathrm{B}\left(\kappa_2 \dots \kappa_n\right)$, fixing κ_1 and s_{max} (which is equivalent to fixing the sum $\sum_{i=2}^{n} \kappa_i$). The minimal value of B from this problem is a lower bound for $\mathrm{B}\left(\kappa_1 \dots \kappa_n\right)$ given κ_1 and s_{max}. The following claim characterizes the solution to problem (116).

Claim 14. If $\kappa_2 \dots \kappa_n$ are an optimal solution to (116), then all but 1 element of $\kappa_2 \dots \kappa_n$ must be equal to either κ_1 or 0.

Proof. We use the the implicit function theorem to calculate the derivative $\frac{dB}{d\kappa_i}$. Write expression (114) as:

$$L = -1 + \frac{2\kappa_1}{2\kappa_1 + B + \sqrt{B^2 + 4\kappa_1^2}} + \sum_{i=2}^{n} \frac{2\kappa_i}{2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}} = 0 \tag{117}$$

Differentiate (117) with respect to κ_i, to get:

$$\frac{\partial L}{\partial \kappa_i} : \ \frac{2B\left(B + \sqrt{B^2 + 4\kappa_i^2}\right)}{\sqrt{B^2 + 4\kappa_i^2}\left(B + 2\kappa_i + \sqrt{B^2 + 4\kappa_i^2}\right)^2} \tag{118}$$

which is strictly positive. Differentiate (117) with respect to B to get:

$$\frac{\partial L}{\partial B} : -\sum_{i=1}^{n} \frac{2\kappa_i\left(1 + \frac{B}{\sqrt{B^2+4\kappa_i^2}}\right)}{\left(B + 2\kappa_i + \sqrt{B^2 + 4\kappa_i^2}\right)^2}$$

This is strictly negative, so we have:

$$\frac{dB}{d\kappa_i} = -\frac{\frac{\partial L}{\partial \kappa_i}}{\frac{\partial L}{\partial B}} > 0 \tag{119}$$

Hence, B is strictly increasing in κ_i. Differentiating (118) once again in κ_i, we have:

$$\frac{\partial^2 L}{\partial \kappa_i^2} = -\frac{2B}{\left(B^2 + 4\kappa^2\right)^{\frac{3}{2}}} < 0$$

Hence, $\frac{\partial L}{\partial \kappa_i}$ is strictly decreasing in κ_i. This implies that, if $\kappa_i > \kappa_j$, then $\frac{\partial L}{\partial \kappa_i} < \frac{\partial L}{\partial \kappa_j}$; from (119), this then implies that $\frac{dB}{d\kappa_i} < \frac{dB}{d\kappa_j}$. Now, suppose we have a vector of demand slopes $\kappa_2 \ldots \kappa_n$, such there exists indices i, j such that κ_i, κ_j are strictly between 0 and κ_1, with $\kappa_i \geqslant \kappa_j$. Then, since $\frac{dB}{d\kappa_i} \leqslant \frac{dB}{d\kappa_j}$, the following vector of demand slopes, for sufficiently small δ, lowers the objective value in (116), while maintaining constraint satisfaction:

$$\left(\kappa_2 \ldots \kappa_i + \delta, \ldots \kappa_j - \delta, \ldots \kappa_n\right)$$

In words, this says that, since $\frac{dB}{d\kappa_i}$ is lower for higher κ_i, we can always lower the objective by increasing high values of κ_i and decreasing lower values to keep the sum constant. Thus, if $\kappa_2 \ldots \kappa_n$ are optimal for problem (116), there cannot exist two indices i, j such that κ_i and κ_j are both strictly between 0 and κ_1; thus, the solution to (116) must have all but one element of $\kappa_2 \ldots \kappa_N$ equal to either κ_1 or 0, proving claim 14. □

The bound from claim 14 is not yet useful, as the optimal value from problem (116) does not admit a simple analytical expression. However, the limit of a sequence of relaxations of problem (116) does yield an analytically tractable solution. I construct a sequence of relaxations of problem (116), parametrized by the integer $h \in \{1, 2, \ldots \infty\}$. For any h, let $\kappa_h(x)$ be a function defined on the interval $[2, n+1]$, which is piecewise constant on intervals of length $\frac{1}{h}$. Define $B(\kappa_h(x))$ as:

$$B(\kappa_h(x)) = \left\{ B : -1 + \frac{2\kappa_1}{2\kappa_1 + B + \sqrt{B^2 + 4\kappa_1^2}} + \int_2^n \frac{2\kappa_h(x)}{2\kappa_h(x) + B + \sqrt{B^2 + 4(\kappa_h(x))^2}} dx = 0 \right\} \tag{120}$$

Define the hth minimization problem as:

$$\min_{\kappa_h(x)} B\left(\kappa_h\left(x\right)\right)$$
$$\text{s.t. } \int_2^n \kappa_h\left(x\right) = K\left(1 - s_{max}\right),\ 0 \leqslant \kappa_i \leqslant \kappa_1\ \forall i \tag{121}$$

Effectively, (120) splits each $\kappa_2 \ldots \kappa_n$ value in the original problem (116) into h components, which may have different values of κ_i. To see that problem (121) is a strict relaxation of the original optimization problem (116), note that if we constrain $\kappa_h\left(x\right)$ to be constant on the intervals

$$[2,3), [3,4), \ldots [n, n+1)$$

then both the objective and the constraints in problem (121) reduced to the original problem (116). Thus, any value attainable in the original optimization problem is attainable in the relaxed optimization problem, so the optimal value from problem (121), for any h, is a lower bound for the original problem.

The integral in the relaxed problem (120) is simply a weighted sum over a finite number of values of $\kappa_h\left(x\right)$; thus, as in the original problem, the objective function for the relaxed problem is concave in the value of $\kappa_h\left(x\right)$ on any interval. Thus, claim 14 characterizing the solution to the original problem applies to the relaxed problem for any h: at any optimal solution, $\kappa_h\left(x\right)$ must be equal to either 0 or κ_1 on all intervals but one. Taking the limit as $h \to \infty$, the constraint

$$\int_2^n \kappa_h\left(x\right) = K\left(1 - s_{max}\right)$$

implies that $\kappa_h\left(x\right)$ must be equal to κ_1 on a set with measure arbitrarily close to

$$\frac{1}{s_{max}} - 1$$

and, $\kappa_h\left(x\right) = 0$ otherwise, except for an interval which has measure arbitrarily close to 0.

Hence, in the limit as $h \to \infty$, any choice of $\kappa_h(x)$ which minimizes B has:

$$\int_2^n \frac{2\kappa_h(x)}{2\kappa_h(x) + B + \sqrt{B^2 + 4(\kappa_h(x))^2}} dx \to \left(\frac{1}{s_{max}} - 1\right) \frac{2\kappa_1}{2\kappa_1 + B + \sqrt{B^2 + 4\kappa_1^2}}$$

Using (120), in the limit as $h \to \infty$, the minimized value of B thus satisfies:

$$-1 + \frac{2\kappa_1}{2\kappa_1 + B + \sqrt{B^2 + 4\kappa_1^2}} + \left(\frac{1}{s_{max}} - 1\right) \frac{2\kappa_1}{2\kappa_1 + B + \sqrt{B^2 + 4\kappa_1^2}} = 0$$

$$\implies -1 + \frac{1}{s_{max}} \frac{2\kappa_1}{2\kappa_1 + B + \sqrt{B^2 + 4\kappa_1^2}} = 0$$

This can be analytically solved for B, to get:

$$B = \frac{2s_{max} - 1}{s_{max}(s_{max} - 1)} \kappa_1$$

This is thus a lower bound for the optimal value to the original optimization problem (116), hence, this is a lower bound for B given κ_1 and s_{max}. It is a nontrivial lower bound whenever $s_{max} < \frac{1}{2}$; otherwise, it is nonpositive. Since we assumed $\kappa_1 = \kappa_{max}$, this proves claim 13. This lower bound is tight whenever $s_{max} = \frac{1}{n}$ for some integer value of n, as it is exactly the equilibrium value of B when there are n agents with identical demand slopes, $\kappa_i = \kappa$. □

D.1.2 Bounding b_i

Now, we use claim 13 to prove the lower bound, (113), of claim 12. We have:

$$b_i = \frac{2\kappa_i + B - \sqrt{B^2 + 4\kappa_i^2}}{2} \tag{122}$$

This shows that b_i is increasing in B. Thus, we can find a lower bound for the ratio $\frac{b_1}{\kappa_1}$ for the largest agent by plugging in the lower bound for B from claim 13 and solving for b_i.

This gives:

$$\frac{b_1}{\kappa_1} \geqslant \left(1 - \frac{s_{max}}{1 - s_{max}}\right) \tag{123}$$

To bound the ratio $\frac{b_i}{\kappa_i}$ for all other i, I show that the ratio $\frac{b_i}{\kappa_i}$ is decreasing in κ_i, for B fixed. To see this, differentiate (122), fixing B, to get:

$$\frac{db_i}{d\kappa_i} = 1 - \frac{2\kappa_i}{\sqrt{B^2 + 4\kappa_i^2}}$$

This is decreasing in κ_i, so b_i is a concave function of κ_i fixing B, and $b_i = 0$ when $\kappa_i = 0$. Now,

$$\frac{d}{dx}\frac{b_i(\kappa_i)}{\kappa_i} = \frac{\kappa_i b_i'(\kappa_i) - b_i(\kappa_i)}{\kappa_i^2}$$

This is negative if the numerator is negative; using that $b_i(0) = 0$, write the numerator as:

$$\kappa_i b_i'(\kappa_i) - b_i(\kappa_i) = \kappa_i b_i'(\kappa_i) - \int_0^{\kappa_i} b_i'(\kappa)\, d\kappa$$

$$= \int_0^{\kappa_i} b_i'(\kappa_i) - b_i'(\kappa)\, d\kappa$$

Since $b_i'(\kappa_i)$ is a decreasing function, $b_i'(\kappa_i) \geqslant b_i'(\kappa)$ for all $\kappa \leqslant \kappa_i$, hence the integrand is everywhere nonpositive; hence $\frac{d}{dx}\frac{b_i(\kappa_i)}{\kappa_i} \leqslant 0$ for all $\kappa_i > 0$. Thus, $\frac{b_i}{\kappa_i}$ is a decreasing function of κ_i fixing b_i. This gives us:

$$\frac{b_i}{\kappa_i} \geqslant \frac{b_1}{\kappa_1} \geqslant \left(1 - \frac{s_{max}}{1 - s_{max}}\right)$$

which is a nontrivial lower bound when $s_{max} < \frac{1}{2}$. This proves the lower bound (113), and thus proves claim 12.