Competition and Manipulation in Derivative Contract Markets

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Abstract

This paper studies manipulation in derivative contract markets. When traders hedge factor risk using derivative contracts, traders can manipulate settlement prices by trading the underlying spot goods. In equilibrium, manipulation can make all agents worse off. The model illustrates how regulators can define manipulation in a manner distinct from other forms of strategic trading behavior, and shows how the structure of contract and spot markets affect the size of manipulation-induced market distortions.

Keywords: derivatives, manipulation, regulation

JEL classifications: D43, D44, D47, G18, K22, L40, L50

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1 Introduction

This paper studies manipulation in derivative contract markets. If a trader wants to buy or sell exposure to some risk factor, such as US equities, interest rates, volatility, or energy, the simplest way to do so is often to use a derivative contract. Some examples of these contracts are S&P 500 futures, LIBOR and SOFR derivatives, VIX futures and options, and derivatives for commodities such as corn and wheat, base and precious metals, oil, gas and electricity. Derivatives make up some of the world’s largest markets: the total notional size of the interest rate derivatives market alone is over $100 trillion USD.

Derivative contracts are linked to underlying spot markets, either through physical delivery or cash settlement. A cash-settled derivative is simply a contract whose payoff is determined based on some price benchmark, which is constructed based on the trade price of some spot good. If the spot price benchmark accurately reflects some risk factor, contracts settled using the benchmark can be used to trade exposure to this risk factor. For example, a long VIX futures position, at settlement, pays its holder some multiple of the CBOE Volatility Index, which is calculated based on prices of S&P 500 options set in a settlement auction. VIX futures are useful for trading volatility to the extent that the option settlement prices are representative of US equity volatility.

Agents who have positions in both contract and spot markets may have incentives to distort spot markets in order to increase contract payoffs. A trader who is long VIX futures can increase her futures payoffs by buying S&P 500 options at the settlement auction to raise the VIX settlement value. If the trader’s futures position is large, her increased futures profits may outweigh any losses incurred by buying S&P 500 options at elevated prices. If many traders bid this way, however, their bids would add noise to the VIX at settlement, creating nonfundamental risk for all agents holding VIX futures contracts.

Legally, trading spot goods to influence contract payoffs is considered contract market manipulation, and is illegal in the US and many other jurisdictions. Regulators have imposed billions of dollars of fines on market participants for manipulation in the past two decades alone. However, manipulation is poorly understood from the perspective of economic theory. We do not have a precise definition of contract market manipulation,
delineating how it is distinct from other forms of strategic trading behavior in financial markets. We do not know how manipulation affects the welfare of different classes of market participants: whether it is Pareto disimproving, or simply creates transfers from non-manipulators to manipulators. We do not understand what makes contract markets vulnerable to manipulation, or how to empirically measure contract market manipulation risk.

In this paper, I build a simple model of contract market manipulation. The model admits a clean definition of manipulation: it is trading behavior in spot markets, aimed at increasing the payoffs on contract positions. Manipulation can be Pareto-disimproving, harming both hedgers and spot market participants. Manipulation-induced market distortions depend on the size of spot market participants' contract positions, as well as the liquidity and competitiveness of spot markets. The size of manipulation-induced market distortions can be measured, using data which is commonly observed by contract market regulators.

I assume that a large number of risk-averse agents have exogenous exposures to a common risk factor. Agents cannot contract on the risk factor directly, but can trade derivative contracts which are tied to the auction price of a spot good. The spot good is traded by a finite number of spot traders, and I assume these traders' marginal value for the spot good is equal to the risk factor. Thus, if spot traders behaved competitively, the spot market would clear with no trade, the spot auction price would be exactly equal to risk factor, and all agents could perfectly share factor risk using derivative contracts.

In equilibrium, however, first-best risk sharing is not attainable because the spot market is not perfectly competitive. Spot traders have price impact, so they have incentives to trade the spot good in a way that increases their derivative contract payoffs. For example, a spot trader with a long contract position has higher incentives to buy the spot good, since her purchases increase the spot auction price and thus her contract payoffs.

Manipulation causes auction prices to become noisy signals of the risk factor, creating non-fundamental basis risk for all contract holders. Market structure in the spot market determines how vulnerable the contract market is to manipulation. Manipulation-induced distortions are larger when spot markets are less liquid, and agents have lower total storage capacity for spot goods. Holding fixed aggregate storage capacity, distortions are
smaller when spot markets are less concentrated, so competition policy in spot markets can alleviate manipulation risk, even if it does not change the aggregate storage capacity of the market. Distortions depend on the size of spot traders’ contract positions, so contract position limits imposed on spot traders can decrease manipulation risk.

The possibility of manipulation affects spot traders’ contract purchasing decisions. Traders tend to buy less contracts because of manipulation-induced basis risk, but tend to buy more because they anticipate making profits from manipulating spot markets. In equilibrium, spot traders may even over-hedge, purchasing larger derivative positions than their total factor risk exposures.

There are two main effects of manipulation on spot traders’ welfare. Spot traders receive positive transfers in expectation, because they can move prices in favor of their contract positions on average, but they also face increased risk due to settlement price variance created by other spot traders. The negative effects can be strong enough that spot traders, as a group, are worse off in equilibrium, relative to a world in which all agents behaved competitively. In some settings, a regulator could increase all market participants’ welfare by imposing taxes or limits on spot traders’ contract positions.

The conclusions of the model hold under a number of generalizations. I allow spot traders to have asymmetric holding costs for the spot good, to receive arbitrarily distributed inventory shocks for spot goods, and to have arbitrarily distributed derivative contract positions. In the general model, basis risk and manipulation rents can be expressed in terms of the slopes of agents’ auction bid curves, and the variances and covariances of spot traders’ inventory shocks and contract positions. Thus, both metrics can in principle be estimated using market data observed by regulators. Finally, while the baseline model focuses on cash-settled contracts for expositional simplicity, the analysis applies identically for contracts settled through physical delivery.

**Implications for contract market regulators.** Contract market regulators currently do not have a precise economic definition of manipulation. In legal proceedings, manipulation is defined essentially as trading with the intent to create artificial prices, and regulators bring cases against market participants based on evidence, often taken from emails and phone calls, that a trader intentionally created price impact for profit. But traders know that their trades move prices; the role that they play in markets is precisely
to manage and optimize the price impact of their trades. Defining manipulation as intentional price impact is sufficiently broad that many kinds of strategic trade in contract markets could conceivably be classified as manipulation, creating substantial uncertainty for market participants.

This paper’s framework suggests a way to define contract market manipulation, distinctly from other forms of strategic trading. In the paper’s model, price impact has two distinct effects on traders’ behavior. The first, which is present in all imperfectly competitive financial markets, is that traders have incentives to “shade” their bids, providing less liquidity than they would in a competitive market, and passing through less of their inventory shocks to bids to strategically avoid price impact. The second is that spot traders who hold contract positions have incentives to trade in spot markets in ways that increase the payoffs from their contract positions: this force can reasonably be defined as contract market manipulation.

Conceptually, there is a simple policy which eliminates contract market manipulation, without affecting non-manipulative strategic trading: regulators could disallow spot market participants from holding derivative contracts entirely. Spot traders would then have incentives to shade bids, but no incentive to manipulate, since they have no contract positions whose payoffs they can influence. In most cases, banning spot traders from holding derivatives is likely impractical, since spot market participants play an important role in linking spot and contract markets through arbitrage. However, this policy may serve as a useful conceptual benchmark for arguing that a pattern of behavior is manipulative: regulators could ask whether a given trader’s behavior in spot markets could be economically justified, if the trader did not hold contract positions. The model also shows that manipulating agents also have incentives to over-hedge, buying more contracts than necessary to hedge factor risk, in order to increase profits from manipulation. Regulators could use the buildup of very large contract positions as evidence of manipulative intent.

The paper’s results also illustrate how regulators can limit manipulation risk by regulating the structure of contract and spot markets. Contract position limits imposed on spot traders can lower manipulation risk, by lowering spot traders’ incentives to manipulate. Contracts settled using more liquid spot markets are less vulnerable to
manipulation, so regulators should try to ensure that contract markets settle to liquid spot markets, where possible. Manipulation is more likely when spot markets are more competitive, so contract market regulators should work with antitrust and spot market regulators to monitor market structure and limit concentration in spot markets.

**Related literature.** To my knowledge, this is the first paper to analyze the welfare effects of contract market manipulation, microfounding behavior in both the contract and spot markets. Other theoretical papers studying price manipulation include Goldstein and Guembel (2008), Bond, Goldstein and Prescott (2010), and Bond and Goldstein (2015). These papers study settings in which prices are used as inputs for other decisions, generating incentives for market participants to manipulate prices to distort these decisions. Other theoretical papers on price manipulation include Garbade and Silber (1983), Kyle (1983), Paul (1984), Cita and Lien (1992), Pirrong (2001), Lien and Tse (2006), Allen, Litov and Mei (2006), Kyle (2007), and Guo and Prete (2019).

Two earlier papers study contract market manipulation with cash-settled futures contracts. Kumar and Seppi (1992) analyze the equilibrium of a Kyle (1985) common-valued competitive market-maker model with agents holding futures contracts linked to spot prices of an asset. Dutt and Harris (2005) discuss how to set futures position limits for agents to prevent manipulation from dominating prices. Relative to this literature, this paper endogenizes traders’ derivative contract positions, allowing us to analyze the welfare effects of manipulation, and also allows for rich heterogeneity between traders, so the model can realistically be fitted to data.

A number of other papers analyze market and benchmark manipulation empirically. For example, Abrantes-Metz et al. (2012), Gandhi et al. (2015), Bonaldi (2018) analyze LIBOR manipulation, Griffin and Shams (2018) analyze VIX manipulation, Evans et al. (2018) study FX manipulation, Birge et al. (2018) study manipulation in electricity markets, Comerton-Forde and Putninš (2011) and Comerton-Forde and Putninš (2014) study stock price manipulation, and Nozari, Pascutti and Tookes (2019) study the related phenomenon of profitable price impact in convertible bond markets. There is also a large regulatory and legal literature on contract market manipulation, which I discuss briefly in subsection 2.2.

Technically, the spot market auction model of this paper builds on the literature on

Outline. The remainder of the paper proceeds as follows. Section 2 discusses some institutional details of contract market manipulation. Section 3 introduces the baseline model, and section 4 derives the main theoretical results. Section 5 studies various extensions of the model, and shows how to measure manipulation-induced market distortions. Section 6 discusses implications of my findings, and section 7 concludes. Proofs, derivations, and other supplementary material are presented in the appendix.

2 Institutional background

2.1 Derivative contracts, liquidity mismatch, and manipulation

Derivative contracts are often used by traders to hedge risks associated with underlying spot goods. Hedging using derivatives is often preferred to holding spot goods directly, because derivatives are traded with high leverage, and do not have the physical costs associated with transporting and storing spot goods. Derivatives may be cash-settled, in which case they pay their holders based on some price benchmark, or settled by physical delivery, meaning that their holder is entitled to some number of units of the underlying asset. In practice, the two kinds of derivatives function similarly, because the majority of physical-delivery derivatives are closed or “rolled”, and delivery of underlying assets is generally rare.

There is often a liquidity mismatch between contract markets and spot markets: the volume of trade in contract markets is often much larger than the volume of trade in associated underlying spot markets. For example, the Platts Inside FERC Houston Ship Channel benchmark for natural gas prices is based on around 1.4 million MMBtus
of natural gas trades per week\footnote{See “Liquidity in North American Monthly Gas Markets” on the Platts website} open interest in the ICE HSC basis future, which is financially settled based on the Platts benchmark, is more than 75 million MMBtus for many delivery months\footnote{ICE Report Center, End of Day reports for HSC basis futures, as of October 24th, 2018. Open interest is above 30,000 contracts for many delivery months, and the contract multiplier is 2,500 MMBtus.} The Secured Overnight Financing Rate (SOFR), designed to replace USD LIBOR as an interest rate benchmark, is based on average daily volumes of approximately $1 trillion in overnight treasury-backed repo loans\footnote{NY Fed’s Secured Overnight Financing Rate Data} as of 2014, the total notional volume of contracts linked to USD LIBOR was estimated to be greater than $160 trillion\footnote{Financial Stability Board (2018).}

There are two main explanations for this liquidity mismatch. The first is speculation: some market participants may wish to trade exposure to price risk simply to express views on the direction of future prices. Derivatives have lower logistical and capital costs to trade than spot goods, so speculators will tend to prefer trading using derivatives. The second is cross-hedging: if prices of a group of spot goods are very correlated, derivative activity often concentrates in one or a few contracts for liquidity reasons. For example, a gas company located in Texas may choose to hedge risks using Henry Hub, Louisiana gas futures contracts, even if Texas gas futures are available, if the Louisiana contract is more liquid.

As a result of this liquidity mismatch, if a spot market participant holds a large derivative position, she may have incentives to trade spot assets non-fundamentally, in a way that increases the payoffs on her contract position. For example, if a gas trader holds a large position in ICE Houston Ship Channel (HSC) basis futures, she can increase her futures payoff by buying spot gas at HSC to increase the benchmark price. The trader may incur losses on her spot market trades, but could generate much larger profits on her futures position.

### 2.2 Legal background

In the US, manipulation and attempted manipulation of contract markets are illegal under the Commodity Exchange Act of 1936. Regulators have policed contract market...
manipulation aggressively in the last few decades. UBS was fined $15 million by the CFTC in 2018 for attempting to manipulate gold futures contracts. The CFTC and the FERC have fined traders millions of dollars for manipulating oil, gas, electricity, precious metals, and propane derivatives.\(^5\) Fines for financial derivative manipulation are orders of magnitude larger: banks have been fined over $10 billion for FX manipulation,\(^6\) over $8 billion USD for manipulation of LIBOR and other interest rate benchmarks,\(^7\) and over $500 million for manipulation of the ISDAFIX interest rate swap benchmark.\(^8\)

Manipulation law and policy is a contentious topic. The Commodity Exchange Act outlaws manipulation, but does not define it. The CFTC’s operational definition of manipulation essentially states that trades made with the intent to create artificial prices are manipulative.\(^9\) This definition is still vague, and there has been substantial disagreement in both the economic and legal literatures, both on what can reasonably be defined as manipulation under current law, and on how manipulation should be regulated from a social planner’s perspective.

In recent times, the legal literature has largely moved away from the “artificial price” notion, focusing instead on “intent” as the standard of proof for manipulation. \(^{10}\) Perdue (1987) argues that manipulation should be defined as conduct which “would be uneconomic or irrational, absent an effect on market price.” Fischel and Ross (1991), similarly, argue that manipulation can only reasonably be defined based on the intent of the trader.

Regulatory authorities have also largely relied on proof of intent as the primary basis for prosecuting manipulators. Charges are brought based on “smoking gun” evidence that trades were made with the intention of moving benchmark prices, rather than for any fundamental reason. \(^{11}\) Levine (2014) quotes a number of trader chat messages used in FX manipulation lawsuits. Other examples include the CFTC’s lawsuits brought against Parnon Energy, Inc. and others for crude oil manipulation, Energy Transfer Partners, L.P. and others for natural gas manipulation, and Barclays for ISDAFIX manipulation.\(^{12}\)

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\(^6\)Levine (2015)

\(^7\)Ridley and Freifeld (2015)

\(^8\)Leising (2017)

\(^9\)17 CFR Part 180

\(^10\)See the CFTC’s website for Parnon Energy, Inc., Energy Transfer Partners, L.P., and Barclays.
Thus, the legal literature and regulatory authorities have largely taken the stance that, under some circumstances, intentional and acknowledged price impact constitutes illegal manipulation. The precise circumstances under which intentional price impact constitutes illegal manipulation, and how manipulation is distinguishable from other forms of strategic trading in financial markets, are unclear. The need to prove intent also imposes a high bar of proof on regulators, as it is difficult to prosecute manipulators in settings where regulators cannot easily access traders’ communication records. Legally sophisticated market participants could avoid being implicated in manipulation lawsuits, simply by taking care not to acknowledge the price impact of their trades in recorded media.

There is also some disagreement in the legal and economic literatures as to whether manipulation should be regulated at all. [Hieronymus (1977, pg. 328)](Hieronymus_1977) argues that contract market manipulation will not survive under market competition. [Fischel and Ross (1991)](Fischel_Ross_1991) argue that “actual trades should not be prohibited as manipulative regardless of the intent of the trader”, and that market competition is likely to deter manipulation. [Markham (1991)](Markham_1991) similarly proposes to abandon the concept of manipulation, and to instead empower the CFTC to take a broader set of actions to maintain fair and orderly markets.\textsuperscript{11}

3 Model

I assume that agents have exogenous exposures to a common risk factor, and can trade derivative contracts to share factor risk. The model has two core frictions. First, agents cannot contract on the risk factor directly: they must instead contract on the auction price of a spot good, whose price is informative about the risk factor. Second, the spot market is imperfectly competitive, so spot good trades move prices. Together, these two frictions imply that spot traders who buy contracts to hedge factor risk have incentives to trade non-fundamentally in spot markets, to influence the payoffs on their derivative contract positions. This creates nonfundamental risk and prevents agents from perfectly sharing risk.

\textsuperscript{11}A few overviews of market manipulation are [Putninš (2012)](Putnis_2012), [Markham (2014)](Markham_2014), and [Putninš (2020)](Putnis_2020).
There are three kinds of agents. There is a representative pure hedger, who buys derivative contracts, but cannot trade the spot good. There are also \( n > 2 \) identical spot traders, who can trade both derivative contracts and spot goods. I assume that spot traders are negligibly small relative to the hedger. The hedger and the spot trader are risk-averse with CARA utility over wealth, with identical risk aversions: agent \( i \)'s utility if she attains wealth \( W \) is:

\[
U_i(W) = -e^{-\alpha W}
\]  

There is a risk-neutral competitive market maker, who trades derivative contracts, but cannot trade the spot good. The market maker submits a demand schedule, mapping net order quantities, from both the hedger and spot traders, to prices. The market maker cannot distinguish between the hedger’s order, and spot traders’ orders. I will show that, in equilibrium, the market maker will set a constant price for derivatives contracts; let \( p_c \) denote the equilibrium contract price.

Agents play a 5-stage game.

1. Spot traders and the hedger draw their factor risk exposures, \( x_H \) and \( x_1 \ldots x_n \).
2. Spot traders and the hedger submit market orders, \( c_H(x_H) \) and \( c_i(x_i) \), to trade cash-settled derivative contracts with the market maker.
3. The risk factor, \( \psi \), is realized.
4. Spot traders bid to trade the spot good in an auction.
5. Derivative contracts pay all contract holders based on the spot auction price.

In stage 1, the hedger is endowed with \( x_H \) units of a productive asset, and each spot trader \( i \) is endowed with \( x_i \) units of the productive asset, where:

\[
x_H \sim N \left( 0, \sigma_{x_H}^2 \right), \quad x_i \sim N \left( 0, \sigma_{x}^2 \right)
\]

A position \( x_i \) in the productive asset pays its holder \( x_i \psi \) net units of wealth. Thus, the value of each unit of the productive asset depends on a normally distributed risk factor \( \psi \), where:

\[
\psi \sim N \left( \mu_{\psi}, \sigma_{\psi}^2 \right)
\]
Hence, I refer to $x_i$ as agents’ *factor exposures*. The risk factor $\psi$ represents, for example, future oil prices, volatility, stock prices, or interest rates; agents may be positively or negatively exposed to each of these sources of risk. I assume that factor exposures $x_i$ are privately observed. I assume the sizes of spot traders’ factor risk exposures $x_i$ are comparable with each other, but spot traders’ positions are collectively infinitesimal, compared to the hedger’s position $x_H$. For analytical convenience, I normalize wealth by assuming that an agent (spot trader or hedger) who has factor exposure $x_i$ is also endowed with $-x_i \mu_\psi$ units of wealth, so that the expectation of all agents’ wealth is 0, regardless of $x_i$.

In stage 2, agents can trade derivative contracts to hedge their factor exposures. In the baseline model, I assume contracts are cash-settled. This is purely for expositional simplicity: in subsection 5.4 I show that all outcomes are identical if contracts are instead settled by physical delivery of the spot good. I use $c_i$ to denote the contract position of agent $i$, and I assume that agents’ contract positions are private information. I assume it is costless to hold derivative contracts. Derivative contracts are useful because agents can purchase them prior to the realization of $\psi$, so contracts allow agents to hedge factor risk exposures. I will show that, in equilibrium, the market maker will set a constant price for contracts, $p_c$, independent of order flow.

In stage 3, the risk factor $\psi$ is drawn and commonly observed by agents. In stage 4, spot traders bid to trade the spot good in a uniform-price double auction. In stage 5, the auction clearing price $p$ is used to settle agents’ derivative contracts: that is, each agent is paid $c_i p$. In the spot market, the $n$ spot traders trade a single homogeneous spot good. A spot trader who purchases $z_i$ net units of the spot good attains a monetary payoff:

$$\psi z_i - \frac{1}{2 \kappa} z_i^2$$  \hspace{1cm} (2)

The $\psi z_i$ component of (2) implies that the marginal value of spot traders for the spot good depends on the risk factor, $\psi$. The quadratic component, $\frac{1}{2 \kappa} z_i^2$, implies that spot traders have decreasing marginal values for the spot good; this can be thought of as a storage or holding cost for the spot good. For physical goods such as oil or gas, the quadratic term might represent physical storage and infrastructure costs; for financial assets such as FX, interest rate swaps, or repo, these costs may correspond to capital or balance sheet
costs. Alternatively, these costs may arise from anticipated price impact from liquidating
spot positions in the future. The parameter $\kappa$ is a measure of agents’ storage or holding
capacity for the asset: when $\kappa$ is larger, spot traders are more willing to take on large spot
good positions, and in equilibrium the price impact of spot trades will be smaller.

Combining all terms, a spot trader’s total wealth, over her factor exposures and outcomes in both
the contract and spot markets, is:

$$W_{\text{spot}}(x_i, c_i, z_i, p, \psi) = \psi x_i - \mu \psi x_i - p c_i + p c_i + z_i \psi - \frac{1}{2} \kappa z_i^2 - p z_i$$  (3)

The pure hedger’s total wealth is:

$$W_{\text{hedger}}(x_H, c_H, p, \psi) = \psi x_H - \mu \psi x_i - p c_H + p c_H$$  (4)

### 3.1 Benchmarks

To illustrate how the two frictions, noncontractibility and imperfect competition, prevent agents
from perfectly sharing risk, we analyze two benchmark cases in which the first-best outcome is
attainable: when agents can directly contract on $\psi$, and when contracts settle to spot market
prices, but spot market participants bid competitively.

**Contracting directly on $\psi$.** Agents could perfectly share risk if they could purchase
contracts tied directly to the risk factor – that is, contracts which pay $\psi$ per contract in stage 5.
In the first period, the market maker would set the price of these contracts equal to their expected
payoff:

$$p_c = E[\psi] = \mu \psi$$

The market maker is willing to buy or sell an infinite amount of contracts at this price,
since the hedger and spot traders’ endowments and contract purchases are uninformative
about $\psi$. An agent with contract position $c_i$ would receive total wealth:

$$x_i \psi - x_i \mu_{\psi} + \psi c_i - p_c c_i$$

Factor exposure  Contract payoff  Contract price

At price $p_c = \mu_{\psi}$, spot traders and the hedger would all choose to hedge perfectly, setting $c_i = -x_i$ and $c_H = -x_H$. Thus, all agents would be able to perfectly hedge, transferring all their factor risk exposures to the risk-neutral market maker.

The first friction in the model is that agents cannot contract directly on $\psi$. This is realistic: in practice, traders cannot contract on abstract risk factors, such as oil prices, volatility, stock prices, or interest rates, directly. Instead, traders can buy and sell contracts which are settled based on price benchmarks set in markets for spot goods, such as the Brent or WTI oil price indices, the VIX, the S&P 500 and Russell 2000, and LIBOR or SOFR.

**Competitive spot market bidding.** Even if agents cannot contract on $\psi$ directly, first-best risk sharing is possible with contracts settled using spot market prices, if spot traders behaved competitively in the spot market, bidding as if their trades did not affect prices. Each spot trader’s marginal value of the spot good is the derivative of (2) with respect to $z_i$:

$$\psi - \frac{z_i}{\kappa}$$

Suppose each spot trader bid in the auction as if she faced a perfectly elastic residual supply curve; that is, suppose each spot trader $i$ believes that the auction clears at some random price $p$, which does not depend on $i$’s quantity traded. $i$ then purchases goods up to the point where her marginal utility is equal to the auction price; this is implemented by submitting a bid curve equal to the inverse of (5), her marginal value for the spot good:

$$z_{Bi}(p) = \kappa (\psi - p)$$

The spot auction would then always clear at price $p = \psi$. There would be no trade of the spot good; this is efficient, since traders’ valuations for the spot good are identical. Anticipating that $p = \psi$, spot traders and the hedger would buy contracts to fully hedge their factor exposures, so agents would be able to perfectly share factor risk. In this
benchmark case, the auction plays no allocative role: its role is to elicit spot traders’ marginal utility for the asset, using it to settle contracts.

The second friction in the model is that the spot market is imperfectly competitive. Spot traders are strategic, and recognize that their spot good trades move the settlement price $p$. Suppose a spot trader has a negative factor exposure $x_i$, and buys a positive contract position $c_i$ to hedge her factor risk exposure; in the spot market, the trader has an incentive to increase her bid in the spot auction, in order to increase $p$ and thus her contract payoffs. This makes $p$ a noisy signal of $\psi$, and prevents perfect risk sharing. I analyze how these incentives play out in equilibrium in the following section.

3.2 Discussion of model assumptions

Assumptions on the contract market. As in Kyle (1985), I assume that there is a risk-neutral market maker, and that spot traders’ factor risk exposures, and thus their contract purchases, are infinitesimal relative to the hedger. As I show in appendix A.7, these assumptions imply that market makers will set the price of contracts equal to $\mu_\psi$, the expectation of the risk factor $\psi$. Intuitively, since spot traders are infinitesimal, their contract purchases are perfectly hidden within the representative hedger’s order flow, and thus net order flow is totally uninformative about the settlement price $p$. The assumption that spot traders are small, so contract purchases have no price impact, is likely a reasonable approximation for many (but not all) contract markets: subsection 2.1 shows that contract markets are often much larger than spot markets.

I assume that there is no cost to hold contract positions. This is a reasonable approximation in many settings: derivatives are useful because they are highly leveraged financial assets, so they have no physical holding costs, and low capital costs relative to spot goods. Agents’ contract positions are still limited, however, by their ability to bear factor risk: an agent who purchases a very large contract position holds a very large exposure to $\psi$.

Contract purchases may be motivated by speculation rather than hedging. In appendix B.4 I assume agents have heterogeneous beliefs about the mean $\mu_\psi$ of the risk factor; this leads agents to purchase nonzero contract positions, because agents think contracts are
mispriced in the first period. The implications for manipulation in the spot market are unchanged.

Finally, while the model of the contract market is quite stylized, assumptions about the contract market only matter for the results in section 4 regarding spot traders’ contract purchasing decisions and welfare. As I discuss in section 5, spot traders’ contract positions at settlement are sufficient statistics for estimating their incentives to manipulate spot markets. Details of why and how spot traders entered into their contract positions do not affect these calculations. Thus, my measure for manipulation-induced welfare losses for hedgers are valid in a variety of different models of the contract market.

**Assumptions on the spot market.** The baseline spot market model is intentionally stylized, in order to illustrate the main forces at work in the model. In section 5, I allow spot traders to have heterogeneous spot good storage capacities, inventory shocks, and arbitrarily distributed contract positions.

Throughout the paper, I take the number of spot market participants, \( n \), and their spot good holding capacities, \( \kappa \), as exogeneous; I am essentially analyzing contract market manipulation holding fixed spot market structure. Spot market structure can change, as agents enter, exit, and adjust their holding costs, but these changes tend to be costly and take time. For physical spot goods, such as oil and gas, holding costs depend on costly infrastructure such as pipelines and storage facilities; for financial assets, many benchmarks are set in inter-dealer markets, which do not allow free entry. Thus, the assumption that spot market structure is essentially fixed in the short run seems reasonable in many settings; however, the metrics I propose would need to be updated over time, as spot market structure changes.

A related concern is that, if the contract market is very vulnerable to manipulation, market participants could hedge factor risk by holding spot goods directly. However, as subsection 2.1 discusses, market participants would generally prefer to hedge using derivatives, because holding spot goods involves higher logistical and capital costs. If manipulation makes derivative contracts sufficiently unattractive that market participants choose to hedge using spot goods, despite their higher costs, there is a social welfare loss, because market participants could share risk at lower cost using derivatives in the first-best outcome.
I assume that the representative hedger cannot trade the spot good. It is equivalent to assume that the hedger can trade the spot good, but has no storage capacity: effectively, the hedger has $\kappa = 0$. This implies that the hedger has perfectly inelastic demand for the spot good, and cannot adjust her purchases in response to prices. For example, an airline or chemicals plant may regularly purchase oil or other commodities as inputs to production, but these inputs may be very inelastic in the short run. This implies that the hedger must submit a perfectly inelastic demand curve in the spot market, and has no ability to manipulate by adjusting her spot trades depending on her contract position.

I assume agents optimize independently of each other; however, in the model, spot traders have incentives to collude. A spot trader who increases settlement prices generates profits for all other traders with long contract positions; thus, colluding spot traders, who coordinate their spot market bids to maximize joint profits, would manipulate more per unit contract than independently optimizing traders. This is an interesting direction to explore, but I leave this to future work.

4 Equilibrium

Proposition 1 describes equilibrium values of spot traders’ bid curves, auction prices, contract purchases, and expected utility.

**Proposition 1.** For any $\alpha, \sigma^2, \kappa, \sigma^2_x, n$, there is a unique equilibrium, in which spot traders’ spot market bids and contract purchasing strategies are linear and symmetric across agents. In stage 1, the market maker sets contract price $p_c = \mu_\psi$. Spot traders’ bid curves in the spot market are:

$$z_{Bi}(p; c_i, \psi) = \frac{1}{n-1} c_i - \frac{n-2}{n-1} \kappa (p - \psi)$$  \hspace{1cm} (7)

Spot auction prices are:

$$p - \psi = \frac{\sum_{i=1}^{n} c_i}{n(n-2)\kappa} \sim N\left(0, \frac{\sigma^2_c}{n(n-2)^2 \kappa^2}\right)$$  \hspace{1cm} (8)

Spot traders’ contract positions $c_i(x_i)$ are linear in traders’ factor exposures $x_i$, so spot traders’
contract positions are normally distributed with mean 0 and variance $\sigma^2_c$, where:

$$c_i(x_i) = -tx_i, \quad \sigma^2_c = t^2 \sigma^2_x$$  \hspace{1cm} (9)

where $t$ satisfies:

$$t \equiv \left(1 + \frac{\alpha \sigma^2_n - \kappa}{(\alpha \sigma^2_\psi \kappa) ((n^2 - 2n) \kappa + \alpha \sigma^2_n)}\right)^{-1}$$  \hspace{1cm} (10)

and $\sigma^2_n$ is the unique positive value satisfying:

$$\sigma^2_n = \frac{\sigma^2_n}{n-1} \left(1 + \frac{\alpha \sigma^2_n - \kappa}{(\alpha \sigma^2_\psi \kappa) ((n^2 - 2n) \kappa + \alpha \sigma^2_n)}\right)^{-2}$$  \hspace{1cm} (11)

$$\sigma^2_n > \frac{\kappa - (\alpha \sigma^2_\psi) (n^2 - 2n) \kappa^2}{\alpha \left(1 + \alpha \sigma^2_\psi \kappa\right)}$$  \hspace{1cm} (12)

Spot traders’ expected utility, as a function of $t$, is:

$$\sqrt{\frac{(n^2 - 2n) \kappa}{\alpha \left(t^2 \sigma^2_x n - 1\right) + (n^2 - 2n) \kappa}} - \sqrt{1 - \alpha \sigma^2_x \left(\alpha \sigma^2_\psi (1 - t)^2 + \frac{\alpha \left(t^2 \sigma^2_x - \kappa\right)}{(\alpha \sigma^2_\psi \kappa) ((n^2 - 2n) \kappa + \alpha \left(t^2 \sigma^2_x - \kappa\right))} t^2\right)}$$  \hspace{1cm} (13)

The hedger’s net contract position is linear in the hedger’s factor exposures $x_H$:

$$c_H(x_H) = -\frac{\sigma^2_\psi}{\sigma^2_\psi + \text{Var}(p - \psi)} x_H = -\frac{\sigma^2_\psi}{\sigma^2_\psi + \frac{\sigma^2_\psi}{n(n-2) \kappa^2}} x_H$$  \hspace{1cm} (14)
The hedger’s wealth has mean 0, and its variance, over uncertainty in $\psi$, $x_H$, and all $x_i$’s, is:

$$\frac{\text{Var}(p - \psi)}{\text{Var}(p - \psi) + \sigma_{\psi}^2} = \left(\frac{\sigma_{\psi}^2}{\frac{\sigma_c^2}{n(n-1)^2} + \frac{\sigma_{\psi}^2}{n(n-2)^2} + \sigma_{\psi}^2} + \sigma_{H}^2\right) \sigma_{\psi}^2 \sigma_{H}^2$$

(15)

### 4.1 Spot market distortions and manipulation-induced basis risk

Spot traders’ equilibrium bids, (7), differ from their competitive bids, (6), in two ways. The first difference, which is known in the double-auctions literature, is that traders shade their bids due to their price impact. The slope of (7) with respect to prices is lower than the slope of (6), by a factor $\frac{n-2}{n-1}$.

The second difference is that traders’ equilibrium bid curves depend on their contract positions $c_i$, even though these do not affect traders’ marginal value for spot goods. Spot traders hold contracts with payoffs that depend on $p$, and traders can move $p$ by trading the spot good, so traders have incentives to trade the spot good to increase contract payoffs. In the context of the model, I refer to this phenomenon as manipulation.

Expression (7) shows that increasing a trader’s contract position by 1 unit causes her to increase her spot good bid curve by $\frac{1}{n-1}$ units; as $n$ increases, contract positions affect bids less. Intuitively, when the spot auction is competitive, agents need to buy more of the spot good to move prices a given amount; the cost of manipulation is higher, so spot traders manipulate less per unit contract that they hold. In the limit as $n$ grows large, expression (7) converges towards (6), and traders’ contract positions have no effect on their bids.

Since auction prices depend on spot traders’ contract positions $c_i$, which depend on traders’ random factor exposures $x_i$, in equilibrium, the auction price $p$ is a noisy signal of the risk factor $\psi$. The difference, $p - \psi$, is normally distributed, and expression (8) characterizes its variance:

$$\text{Var}(p - \psi) = \frac{\sigma_c^2}{n(n-1)^2 \kappa^2}$$

(16)

I call this variance manipulation-induced basis risk. Basis risk is determined by three factors: liquidity, competition, and the size of contract positions. First, basis risk decreases to 0 as
the spot market becomes infinitely liquid, \( n \kappa \to \infty \), holding fixed \( \sigma_c^2 \). Since \( \kappa \) represents the storage capacity of an individual agent, \( n \kappa \) can be thought of as the aggregate storage capacity of the market. \( n \kappa \) is also approximately equal to the slope of aggregate auction demand, up to a bid-shading factor\(^{12}\). When \( n \kappa \) is large, holding \( \sigma_c^2 \) fixed, the price impact of manipulative spot trading decreases towards 0, so prices converge to \( \psi \).\(^{13}\)

Second, basis risk decreases as markets become more competitive. Price variance decreases to 0 if \( n \to \infty \), holding fixed \( n \kappa \) and \( \sigma_c^2 \); that is, basis risk decreases towards 0 if we increase \( n \), holding fixed the aggregate storage capacity in the market, as well as the size of individual agents’ contract positions.\(^{14}\) The intuition for this is that, from (7), the coefficient on \( c_i \) in agents’ equilibrium bids is \( \frac{1}{n-1} \): in more competitive markets, agents’ contract positions affect their bidding decisions less, and thus create less price variance, even if aggregate storage capacity is fixed. This implies that competition policy in spot markets can alleviate manipulation: blocking mergers, and splitting large spot traders into smaller entities, can decrease manipulation-induced basis risk, even if it does not change the market’s total storage capacity.

Finally, basis risk depends on the size of traders’ contract positions, \( \sigma_c^2 \): spot traders manipulate more, and create more basis risk, when their contract positions are larger. This implies that policies such as regulatory position limits imposed on spot traders can lower manipulation-induced basis risk.

### 4.2 Spot traders’ contract positions

Expression (9) of proposition 1 shows that, in the unique equilibrium of the model, agents’ optimal contract purchases \( c_i \) are linear in their factor exposures \( x_i \). The coefficient of proportionality, defined as \( t \) in expression (10), describes how aggressively agents are

\(^{12}\)Formally, the slope of aggregate auction demand is \( n \) times an individual bidder’s bid slope, so it is \( \frac{n-2}{n-1} n \kappa \).

\(^{13}\)Note that, while prices become competitive as \( n \kappa \to \infty \), spot market quantities do not necessarily become competitive. From (7), the coefficient on contract positions in agents’ bids is \( \frac{1}{n-1} \), independent of \( \kappa \). If \( \kappa \) increases holding \( n \) fixed, prices converge to \( \psi \), but spot market bids and trade quantities are still distorted in the limit.

\(^{14}\)Note that \( \sigma_c^2 \) is the variance of an individual spot trader’s contract position: as \( n \) increases, the total size of contract positions across all agents increases.
hedging; that is, how many contracts agents purchase per unit of their factor exposures.

If all traders bid competitively in spot markets, traders would perfectly hedge, setting \( t = 1 \). In equilibrium, \( t \) can be greater or smaller than 1. On the one hand, spot traders anticipate that others will manipulate, making spot prices noisy signals of \( \psi \), decreasing their incentives to buy contracts to hedge factor risk. On the other hand, spot traders anticipate that they can profit on average from their contract positions, since they can move spot prices in their favor, increasing their incentives to buy contracts.

The second force may dominate the first, so we can have \( t > 1 \) in equilibrium. In this case, spot traders will “over-hedge”, buying contract positions which are larger than their original factor risk exposures \( x_i \). In other words, agents may purchase contract positions so large that they actually increase their exposures to factor risk, because of anticipated profits from moving contract settlement prices in their favor. Agents do not purchase infinitely large contract positions, because the increased exposure to basis risk eventually overwhelms any anticipated profits from manipulation.

From expression (10), \( t > 1 \) in equilibrium if:

\[
\kappa > \alpha \sigma^2 \eta
\]

where \( \sigma^2 \eta \), from (11), is the expected variance of residual supply in the spot market. Intuitively, spot traders have incentives to buy large contract positions if storage capacity \( \kappa \) is large, relative to risk aversion \( \alpha \) and the variance of residual supply \( \sigma^2 \eta \). Appendix B.2 shows that the amount of over-hedging in equilibrium can be unboundedly large: there is no upper bound on the equilibrium value of \( t \). Appendix B.3 shows that increasing \( n \), holding other parameters fixed, causes \( t \) to rapidly converge to 1: when there are a large number of market participants, spot traders hedge perfectly, and the market approaches the first-best outcome.

### 4.3 Spot trader welfare

Expression (13) shows spot traders’ expected welfare, over all sources of uncertainty in the model. This allows us to assess whether spot traders are better or worse off in equilibrium, relative to the competitive-bidding benchmark. Expression (13) also allows
us to calculate welfare for values of $t$ other than its equilibrium value. This corresponds to welfare in a limited social planner’s problem, in which the planner can force all spot traders to buy contracts according to some prespecified value of $t$, but cannot influence traders’ behavior in spot markets. This can be thought of as a reduced-form model of various actions contract market regulators can take to limit the size of traders’ contract positions, such as imposing contract position limits.

The three panels of figure 1 show spot traders’ welfare as a function of $t$, alongside the equilibrium and competitive values of $t$ and spot trader welfare, for three different sets of input parameters. A given spot trader’s manipulation generates both positive and negative externalities on other spot traders, so a number of different outcomes are possible: spot traders may gain or lose on average from manipulation, the equilibrium $t$ may be greater or smaller than 1, and spot traders may hedge more or less than is optimal for spot traders’ welfare.

In the left panel, manipulation increases spot traders’ welfare, relative to the competitive benchmark. Spot traders also over-hedge, choosing $t > 1$ in equilibrium. However, spot traders would do even better as a group if they could commit to holding smaller contract positions. Spot traders face a kind of prisoner’s dilemma: manipulation by one spot trader creates a negative externality on other spot traders by increasing basis risk, so all spot traders would prefer a lower value of $t$ than the equilibrium value. A lower value of $t$ would also benefit the hedger, since it would decrease basis risk, so a regulator could create a Pareto improvement by limiting the size of spot traders’ contract positions.

In the middle panel, like the left panel, spot traders would prefer for $t$ to be smaller than its equilibrium value. In addition, spot traders’ equilibrium welfare is below the competitive value of $-1$. As the following subsection shows, the hedger is always worse off in equilibrium relative to the competitive benchmark. Thus, in this example, manipulation is Pareto disimproving relative to the competitive benchmark, decreasing welfare for both spot traders and the hedger. Intuitively, this is because spot traders’ losses from manipulation-induced basis risk outweigh their gains from extracting a transfer.

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15I do not directly analyze position limits because my model requires agents’ contract positions to be Gaussian, so bounds on the size of agent’s contract positions would be intractable. In the context of the model, appendix B.3 shows that the planner could implement any positive value of $t$ in equilibrium by imposing quadratic taxes or subsidies on spot traders’ contract positions, charging spot traders $kc_i^2$ for buying $c_i$ contracts.
Notes. Expected welfare of spot traders, \((13)\), as a function of the hedging aggressiveness parameter \(t\). The green vertical and horizontal lines denote the competitive values of \(t\) and welfare, which are always equal to 1 and -1 respectively. The blue line denotes the unique equilibrium value of \(t\), and the orange line denotes the value of \(t\) which maximizes spot traders’ welfare. The parameters for the left panel are \(n = 3, \alpha = 1, \kappa = 0.5, \sigma^2_\psi = 0.3, \sigma^2_x = 0.5\); for the middle panel, they are \(n = 5, \alpha = 1, \kappa = 0.05, \sigma^2_\psi = 0.9, \sigma^2_x = 1\); for the right panel, they are \(n = 3, \alpha = 1, \kappa = 0.8, \sigma^2_\psi = 1.5, \sigma^2_x = 0.05\).

from the hedger in expectation.

In the right panel, spot traders would actually prefer for \(t\) to be higher than its equilibrium value. This is because manipulation by one spot trader actually creates a positive externality on other spot traders in spot markets: manipulators make non-fundamental trades in spot markets, so other spot traders profit as market makers, buying low and selling high in the spot auction. In this case, spot traders face a kind of coordination problem: all spot traders would be better off if each trader bought more contracts and manipulated more. This would, however, increase basis risk and decrease the hedger’s welfare, so an increase in \(t\) can never be Pareto-improving.

Together, these examples show that manipulation can be Pareto-dominated relative to the competitive benchmark, and regulatory intervention to decrease spot traders’ hedging aggressiveness can be Pareto improving. Appendix B.1 shows additional comparative statics of the model, illustrating how equilibrium outcomes are affected by different model primitives.
4.4 Hedger welfare

Finally, expression (14) shows how manipulation affects the representative hedger’s contract purchasing decisions. The hedger buys less contracts per unit of her risk exposure, because under manipulation, contracts are noisier hedges for factor risk. The optimal hedge is to buy

\[-\frac{\sigma^2_\psi}{\sigma^2_\psi + \text{Var}(p - \psi)}\]

units of contracts, per unit of her factor risk exposure. Intuitively, (17) is the coefficient from regressing the risk factor \(\psi\) on the auction price \(p\). With manipulation-induced basis risk, \(p\) is a noisier signal of \(\psi\), so the hedger optimally hedges less per unit of her factor risk exposure. As a result, the hedger’s wealth has variance:

\[
\frac{\text{Var}(p - \psi)}{\text{Var}(p - \psi) + \sigma^2_\psi \sigma^2_H}
\]

(18)

The ratio \(\frac{\text{Var}(p - \psi)}{\text{Var}(p - \psi) + \sigma^2_\psi}\) is equal to \(1 - R^2\), where \(R^2\) is the coefficient of determination from regressing \(p\) on \(\psi\). When basis risk is 0, \(R^2 = 1\), there is no basis risk, so the hedger can perfectly hedge and face no wealth uncertainty. As basis risk becomes very large, \(R^2 = 0\), so the hedger buys no contracts, and is fully exposed to factor risk.

5 Extensions

In this section, I study a number of extensions of the baseline model. In subsection 5.1, I allow spot traders to have asymmetric storage capacities \(\kappa_i\), as well as asymmetric distributions of contract positions. In subsection 5.2, I allow spot traders to have inventory shocks, which affect their trading decisions as well as contract positions. In subsection 5.3, I show how a regulator could measure manipulation risk in data. Throughout these subsections, I restrict attention to stages 4 and 5 of the game: I study spot traders’ bidding behavior in the spot market, taking spot traders’ contract positions as exogenous random
variables, rather than studying the full multi-stage game. Finally, subsection 5.4 shows that the results of the baseline model also apply to physical delivery contracts.

5.1 Asymmetric agents

Suppose that the wealth of spot market participant i, if she buys $z_i$ net units of the spot good at price $p$, is:

$$W_{\text{spot},i}(c_i, z_i, y_i, p) = \underbrace{pc_i}_{\text{Contract payoff}} + \underbrace{\psi z_i - \frac{z_i^2}{2\kappa_i}}_{\text{Spot good payoff}} - \underbrace{pz_i}_{\text{Spot good price}}$$

Expression (19) generalizes (3) from the baseline model in two ways. First, (19) allows spot market participants to have different holding capacities, $\kappa_i$, for the spot good. Agents with larger $\kappa_i$ have more elastic demand for the spot good; these may be agents who have more storage space for physical spot goods, or lower capital costs for trading financial spot goods. Second, I assume that contract positions $c_i$ have mean 0 and full support, but otherwise can be arbitrarily distributed: in particular, $c_i$ need not be identically distributed across agents.

Compared to (3), I also omit factor exposure and contract price terms from (19). These terms do not depend on $z_i$, so they do not affect traders’ decisions in the spot market. Also, in the spot market, agents’ bids are ex-post best responses over uncertainty in other agents’ bids, so agents’ optimization problem is nonstochastic, and we can solve (19) without the assumption that agents have CARA utility over wealth.

---

16The multi-stage game with asymmetric agents is difficult to solve, because each trader faces a different variance of residual supply in the spot market, $\sigma^2_{\eta_i}$. Solving for equilibrium would involve finding a set of bids for all agents, such that agents’ bids are best responses given spot market bids of other agents, and the variances $\sigma^2_{\eta_i}$, and then bidding and contract purchasing decisions are consistent with all residual supply variances $\sigma^2_{\eta_i}$. This cannot be solved in closed-form, and appears not to add much insight compared to studying the spot market game directly, taking contract positions as exogeneous.

17Full support is needed only to justify the ex-post equilibrium concept: when contract positions have full support, the intercepts of the residual supply curves faced by each trader also have full support, so spot traders’ bid curves are fully determined by requiring them to be best responses for all possible realizations of residual supply. The assumption that each $c_i$ has mean 0 is not needed to solve the model, but implies that $p - \psi$ has mean 0; without it, $p$ would not be equal to $\psi$ in expectation, but the model conclusions would otherwise be unchanged.
Expression \[19\] is an accurate model of spot market behavior under many different assumptions about how the contract market works: for example, as appendix B.4 discusses, spot traders may buy contracts to speculate rather than hedge, or as appendix B.5 discusses, spot traders’ contract purchases may have price impact. A spot trader who holds a long contract position profits from higher spot prices, regardless of why she entered into the contract in the first place. Thus, across many different models, spot traders’ contract positions are an observable sufficient statistic for spot traders’ incentives to manipulate spot markets.

The following proposition characterizes equilibrium bids and prices in the unique equilibrium of the general model.

**Proposition 2.** When spot traders’ wealth is described by \[19\], there is a unique linear ex-post equilibrium, in which \(i\) submits the bid curve:

\[
Z_{B_i}(p; y_i, c_i) = \frac{b_i}{\sum_{j \neq i} b_j} c_i - b_i (p - \psi)
\]  

(20)

The spot auction price is:

\[
p - \psi = \frac{1}{\sum_{i=1}^{n} b_i} \sum_{i=1}^{n} \left[ \frac{b_i}{\sum_{j \neq i} b_j} c_i \right]
\]  

(21)

bid slopes \(b_i\) satisfy:

\[
b_i = \frac{B + 2\kappa_i - \sqrt{B^2 + 4\kappa_i^2}}{2}
\]  

(22)

and \(B = \sum_{i=1}^{n} b_i\) is the unique positive solution to the equation

\[
B = \sum_{i=1}^{n} \frac{2\kappa_i + B - \sqrt{B^2 + 4\kappa_i^2}}{2}
\]  

(23)

Manipulation incentives depend most directly on the coefficient on \(c_i\), \(\frac{b_i}{\sum_{j \neq i} b_j}\), in the bid expression (20). From (22), \(b_i\) is increasing in \(\kappa_i\), so agents with larger storage capacities have greater incentives to manipulate, in the sense that they trade more of the spot good per unit contract that they hold. If the market is fairly competitive, we can obtain a simple approximation to agents’ optimal bid curves, based on agents’ shares of
total storage capacity. Define agent $i$’s capacity share $s_i$ as:

$$s_i \equiv \frac{\kappa_i}{\sum_{i=1}^{n} \kappa_i}$$

(24)

Since $\kappa_i$ represents the spot good storage capacity of agent $i$, $s_i$ can be thought of as $i$’s share of total storage capacity in the spot market. Define $s_{\text{max}}$ as the largest agent’s capacity share:

$$s_{\text{max}} \equiv \max_i s_i$$

The following proposition derives bounds on the $c_i$ coefficients, $\sum_{j \neq i} b_j$, as functions of $s_i$ and $s_{\text{max}}$.

**Claim 1.** When $s_{\text{max}} < \frac{1}{2}$, we have:

$$s_i \leq \sum_{j \neq i} \frac{b_j}{b_i} \leq \left(1 + \frac{s_{\text{max}}}{1 - 2s_{\text{max}}}\right) s_i$$

(25)

Claim 1 implies that the coefficient on $c_i$ in $i$’s bid in equilibrium is within a factor

$$\left(1 + \frac{s_{\text{max}}}{1 - 2s_{\text{max}}}\right)$$

of her capacity share, $s_i$. Thus, when $s_{\text{max}}$ is small, $c_i$ coefficients are approximately equal to capacity shares. Intuitively, when markets are relatively competitive, an agent who constitutes roughly 10% of the market’s storage capacity will increase her spot market bids by approximately 10% of the size of her contract position. The factor $s_i$ thus generalizes the factor $\frac{1}{n-1}$ in bids, in (7) of the baseline model.

If all agents’ contract positions have equal variance, $\sigma_c^2$, we can also derive bounds for manipulation-induced basis risk. Define the capacity HHI (Herfindahl-Hirschman index) as:

$$\text{HHI} \equiv \sum_i s_i^2$$

(26)

The HHI is a standard measure of market concentration used in antitrust analysis. When agents are symmetric, $\kappa_i = \kappa$, the HHI is simply $\frac{1}{n}$.

**Claim 2.** When $s_{\text{max}} < \frac{1}{2}$, and agents’ contract positions have the same variance $\sigma_c^2$,
manipulation-induced basis risk satisfies:

\[
\frac{\sigma_c^2}{B^2} \text{HHI} \leq \text{Var} (p - \psi) \leq \left( 1 + \frac{s_{\max}}{1 - 2s_{\max}} \right)^2 \frac{\sigma_c^2}{B^2} \text{HHI}
\]  

(27)

Claim 2 generalizes the comparative statics discussion in subsection 4.1 to the asymmetric model. Basis risk depends on the size of contract positions, \( \sigma_c^2 \); the slope of aggregate auction demand, \( B \); and market concentration, \( \text{HHI} \). As in subsection 4.1, holding fixed \( \sigma_c^2 \), price variance decreases towards 0 as \( B \) becomes very large, so markets are very liquid. Price variance also decreases towards 0 as \( \text{HHI} \) gets very small, so markets are very competitive.

Next, suppose traders’ contract positions may have different variances: let \( \sigma_{ci}^2 \) represent the variance of \( i \)'s contract position. Basis risk is then:

\[
\text{Var} (p - \psi) = \frac{1}{B^2} \sum_{i=1}^{n} \left( \frac{b_i}{\sum_{j \neq i} b_j} \right)^2 \sigma_{ci}^2
\]

(28)

Expression (28) states that basis risk is the weighted sum of contract position variances, \( \sigma_{ci}^2 \), with weights \( \left( \frac{b_i}{\sum_{j \neq i} b_j} \right)^2 \). These weights are larger for traders with larger storage capacity \( \kappa_i \). Intuitively, agents with larger spot good storage capacity will trade more in the spot market per unit contract they hold, so basis risk will be higher when agents with large storage capacity hold large contract positions. This suggests that regulators may actually want to impose stricter position limits on market participants with larger spot good storage capacities.

5.2 Inventory shocks

Next, we add inventory shocks to the model. Assume spot traders’ wealth is:

\[
W_{\text{spot},i} (c_i, z_i, y_i, p) = \underbrace{pc_i}_{\text{Contract payoff}} + \underbrace{\psi z_i - \frac{(z_i + y_i)^2}{2\kappa_i}}_{\text{Spot good payoff}} - \underbrace{pz_i}_{\text{Spot good price}}
\]

(29)
Expression (29) assumes that agents enter the spot market with some existing inventory position, \( y_i \), in the spot good. As in the previous subsection, I assume that each \( y_i \) has mean 0 and full support, but otherwise can be arbitrarily distributed. Inventory shocks imply that agents will trade the spot good even if they do not hold contract positions, so that not all trade volume in the spot market is caused by manipulation. The following proposition characterizes equilibrium bids and prices in equilibrium with inventory shocks.

**Proposition 3.** When spot traders’ wealth is described by (29), there is a unique linear ex-post equilibrium in the general model, in which \( i \) submits the bid curve:

\[
zb_i(p; y_i, c_i) = -\frac{b_i}{\kappa_i} y_i + \frac{b_i}{\sum_{j \neq i} b_j} c_i - b_i (p - \psi)
\]  

The spot auction price is:

\[
p - \psi = \frac{1}{\sum_{i=1}^{n} b_i} \sum_{i=1}^{n} \left[ -\frac{b_i}{\kappa_i} y_i + \frac{b_i}{\sum_{j \neq i} b_j} c_i \right]
\]  

and bid slopes \( b_i \) satisfy (22) and (23) of Proposition 2.

First, I specialize proposition 3 to the case in which agents are fully symmetric.

**Corollary 1.** Suppose that \( \kappa_i = \kappa \), \( \text{Var}(y_i) = \sigma_y^2 \), \( \text{Var}(c_i) = \sigma_c^2 \) for all \( i \), and all \( y_i \) and \( c_i \) are independent. Spot traders’ equilibrium bids are:

\[
z_{Bi}(p; c_i, y_i) = -\frac{n - 2}{n - 1} y_i + \frac{1}{n - 1} c_i - \frac{n - 2}{n - 1} \kappa (p - \psi)
\]  

Price variance is:

\[
\text{Var}(p - \psi) = \frac{\sigma_y^2}{n\kappa^2} + \frac{\sigma_c^2}{n(n - 2)^2 \kappa^2}
\]  

18Inventory shocks \( y_i \) can alternatively be interpreted as preference shocks for the spot good: expanding the quadratic term in (29), we have:

\[
-\frac{z^2}{2\kappa_i} - \frac{y_i z}{\kappa_i} - \frac{y_i^2}{2\kappa_i} \text{.}
\]  

Ignoring the constant \( \frac{y_i^2}{2\kappa_i} \) term, \( y_i \) simply linearly shifts \( i \)’s marginal value of the spot good.
Expressions (32) and (33) are very similar to the expressions for bids and prices in the baseline model, in proposition 1; the only difference is the inclusion of $y_i$ terms. As $n$ increases, the $y_i$ coefficient increases towards 1, whereas the $c_i$ coefficient decreases towards 0. Intuitively, $y_i$ reflects a fundamental, inventory- or utility-driven component of bids, whereas $c_i$ reflects a nonfundamental component, reflecting agents’ preferences over prices, driven by their contract positions. When $n$ is higher and agents have less price impact, their bids reflect fundamentals more, and preferences over prices less.

Another implication of (32) is that, in relatively competitive markets, the coefficient on $c_i$ tends to be much smaller than the coefficient on $y_i$: an agent who receives a unit inventory shock changes her spot bid curve by approximately one unit, whereas an agent with a unit contract shock changes her spot bid by approximately $\frac{1}{n}$ units. Thus, when $n$ is large, most of the variation in agents’ bids (and thus settlement prices) will be driven by inventory shocks, even if the variance of $c_i$ is much larger than the variance of $y_i$. Quantitatively, from (33), inventory positions and contract positions contribute equal amounts to settlement price variance when:

$$\sigma_c = (n - 2) \sigma_y$$  \hspace{1cm} (34)

As subsection 2.1 discusses, contract markets are often much larger than spot markets in practice. It is a puzzle why this liquidity mismatch can be sustained, without creating very large manipulation incentives for spot traders. Expression (34) provides a simple answer: contract positions can be much larger than inventory positions, as long as spot markets are sufficiently competitive.\(^{19}\)

When agents have different $\kappa_i$ values, from (30) of proposition 3 and (22) of proposition 2 agents with higher values of $\kappa_i$ – larger capacity relative to the market – will submit bid curves with higher $p_i$ coefficients, lower $y_i$ coefficients, and higher $c_i$ coefficients.\(^{20}\) Intuitively, these agents submit bid curves which are steeper with respect to prices, so

\(^{19}\)Expression (34) could be used as a simple rule-of-thumb for evaluating whether a given market is vulnerable to manipulation: regulators need only check whether the total size of spot traders’ outstanding contract positions is more than $(n - 2)$ times larger than the total size of their inventory positions. This test only formally works under strong symmetry assumptions, but may be a reasonable first-pass when limited data are available.

\(^{20}\)I formally show that the coefficient on $y_i$, $\frac{b_i}{\kappa_i}$, is decreasing in $\kappa_i$, in appendix D.1.2.
they absorb more of other agents’ inventory shocks. They shade their inventory positions more, trading less per unit of their inventory shocks $y_i$. They manipulate more, trading more per unit of their contract positions $c_i$. As a result, expression (31) shows that equilibrium prices depend on a weighted sum of $c_i$ and $y_i$: the weights on $y_i$ are lower, and the weights on $c_i$ are higher, for agents with larger $\kappa_i$ values.

5.3 Measurement

Next, I show how basis risk can be estimated by a regulator. To ensure that basis risk is finite, we assume that all $y_i$ and $c_i$ have finite variances and covariances. Expression (31) shows that auction prices are linear in agents’ inventory and contract positions: this implies that the moments of auction prices are simple functions of the moments of $y_i$ and $c_i$, allowing us to derive a simple expression for basis risk. As in proposition 3, we define

$$B \equiv \sum_{i=1}^{n} b_i$$

as the sum of agents’ bid slopes $b_i$. Define the coefficient vectors $k_c, k_y$ as:

$$k_c = \begin{pmatrix} b_1 \\ \frac{b_1}{\sum_{j \neq 1} b_j} \\ \vdots \\ \frac{b_n}{\sum_{j \neq n} b_j} \end{pmatrix}, \quad k_y = \begin{pmatrix} b_1 \\ \kappa_1 \\ \vdots \\ \kappa_n \end{pmatrix}$$

Define the covariance matrices of agents’ contract position, $\Sigma_{cc}$, agents’ inventory positions, $\Sigma_{yy}$, and the inventory-contract covariance matrix $\Sigma_{yc}$, respectively, as matrices with elements:

$$\Sigma_{cc}(i, j) = \text{Cov}(c_i, c_j), \quad \Sigma_{yy}(i, j) = \text{Cov}(y_i, y_j), \quad \Sigma_{yc}(i, j) = \text{Cov}(y_i, c_j)$$

Proposition 4. In the general model of proposition 3, basis risk is:

$$\text{Var}(p - \psi) = \frac{1}{B^2} \left( k'_y \Sigma_{yy} k_y - 2 k'_y \Sigma_{yc} k_c + k'_c \Sigma_{cc} k_c \right)$$

(35)

Proposition 4 shows that, in order to estimate basis risk, regulators need to observe data on agents’ bid slopes in spot markets, and the variances and covariances of agents’ inventory and contract positions.
Bid slopes. In order to estimate $k_c$ and $k_y$, the econometrician must estimate $b_i$ for all agents, that is, the slope of each agent’s equilibrium bid curve with respect to prices. The coefficient vectors $k_c$ and $k_y$ are functions of both $b_i$ and agents’ spot holding capacity $\kappa_i$; however, $\kappa_i$ can be recovered from the vector of $b_i$ values, and the aggregate bid slope $B$, by inverting (22) of proposition 3. For benchmarks which are set in auctions, bid slopes could be estimated directly from agents’ auctions bids. If bidding data are not available, an alternative approach would be to model or estimate agents’ spot holding capacities, $\kappa_i$, and then calculate implied bid slopes $b_i$ by solving for equilibrium in proposition 3.

Inventory-contract variance and covariances. The econometrician must also estimate the variance-covariance matrices of $c_i$ and $y_i$. In many markets, regulators have access to detailed and high-frequency data on agents’ positions in both spot and derivative markets, so the variance and covariance matrices could be estimated using historical sample moments of $c_i$ and $y_i$. If $c_i$ is not observable, the econometrician could derive bounds on $\Sigma_{cc}$ based on contract position limits which are imposed by exchanges or regulators. If $y_i$ is not observable, under some distributional assumptions, the econometrician could estimate or bound $\Sigma_{yy}$ using the observed volume of trade, since more variable inventory shocks imply higher trade volumes in spot markets.

In appendix C.5 I define another measure of manipulation-induced distortions, manipulation rents, which are equal to the expected profits that spot traders as a group earn, due to their ability to move spot market prices. I show how to estimate manipulation rents in the general asymmetric model. In the model of the paper, basis risk is a sufficient statistic for the hedger’s welfare losses; however, manipulation rents may also be a useful metric to quantify spot traders’ aggregate potential profits from manipulation.

5.4 Physical delivery contracts

Thus far, we have assumed all contracts are cash-settled. In this subsection, based on an equivalence result in Kyle (2007), I show that the analysis is essentially unchanged if contracts are instead settled by physical delivery.

---

21 The covariance matrices are still high-dimensional, so finite-sample error may be a concern; this can be solved using a variety of econometric methods, such as low-dimensional factor models for agents’ contract and inventory covariances.
Suppose contracts are settled by physical delivery: each unit of the derivative contract entitles agents to receive a unit of the spot good during the settlement auction. All agents – the hedger, the market maker, and the \( n \) spot traders – submit bid curves in the spot auction, taking into account the units that they are entitled to from their contract positions. Market clearing in the spot market thus requires the sum of each agents’ net demand – consisting of promised units from agents’ contract positions, plus the amount that agents bid for – to sum to 0. Let
\[
c_H, z_{B,H}(p), c_{MM}, z_{B,MM}(p)
\]
respectively denote contract positions and spot market bids from the hedger, \( H \), and the market maker, \( MM \). Let \( q_H \) be the net amount of the spot good that the hedger purchases, summing over her contract position and her spot good trades:
\[
q_H \equiv c_H + z_H
\]
and let \( q_{MM} \) be defined likewise. We have assumed that the hedger and market maker cannot actually make or take delivery of the spot good, since they have no storage capacity. Hence, both parties must have net trade quantity equal to 0, so they are forced to submit market orders – perfectly inelastic demand curves – to sell exactly the amount of the spot good that they are entitled to in contracts:
\[
z_{B,H}(p) = -c_H, \quad z_{B,MM}(p) = -c_{MM} \tag{36}
\]
In practice, (36) can be interpreted as saying that the hedger and the market maker exit their contract positions just before settlement. The market maker’s wealth is thus:
\[
W_{MM}(c_{MM}) = p_c c_{MM} - p (-c_{MM}) \tag{37}
\]
As in the baseline model, (37) implies that the market maker purchases contracts at \( p_c \), and are paid the spot price \( p \). Thus, risk-neutral competitive market makers will set
\[ p_c = E[p \mid c_{\text{MM}}], \text{ as in the baseline model. The hedger's wealth is:} \]

\[
W_{\text{hedger}}(x_H, c_H, p, \psi) = \psi x_H - \mu_p x_i - p_c c_H - p(-c_H) \tag{38}
\]

Expression (38) is identical to (4) in section 3, so physical delivery contracts are identical from the perspective of the hedger. If \( p = \psi \), these contracts are effective for hedging, because the hedger buys them at \( p_c \) and sells the resultant inventory at \( \psi \). Effectively, the hedger hedges by pre-purchasing a promise to deliver at \( \mu_p \), and then simply selling at the realized price \( \psi \). If \( p \neq \psi \), the hedger hedges imperfectly, because the price that the hedger gets by liquidating her spot inventory does not reflect \( \psi \) perfectly.

**Spot traders.** Spot traders essentially start in the spot market with an inventory position \( y_i = c_i \) in the spot good. Their wealth is thus:

\[
W_{\text{spot}}(x_i, c_i, z_i, p, \psi) =
\]

\[
\psi x_i - \mu_p x_i - p_c c_i + (z_i + c_i) \psi - \frac{1}{2k} (z_i + c_i)^2 - p z_i
\]

Expanding, and dropping terms that do not depend on \( c_i \) or \( z_i \), we have:

\[
c_i \psi + z_i \psi - \frac{1}{2k} \left( z_i^2 + 2z_i c_i + c_i^2 \right) - p z_i \tag{39}
\]

Expression (39) shows that the hedger and the market maker make and take no deliveries of the spot good. Thus, in order for spot markets to clear, the total quantities \( z_i \) purchased by spot traders, plus their total contract positions \( c_i \), must sum to 0:

\[
\sum_i z_i (p; c_i, \psi) + c_i = 0
\]

We can define spot traders’ net demand \( q_i \) as the sum of their spot purchases \( z_i \), and their contract positions \( c_i \):

\[
q_i (p; c_i, \psi) \equiv z_i (p; c_i, \psi) + c_i \tag{40}
\]

Defined thus, when \( z_i = -c_i \), trader \( i \) has zero net demand: practically, she closes all her
contract positions just before settlement. When \( z_i = 0 \), trader \( i \) takes delivery of her entire contract position \( c_i \). When \( z_i \) is between \( 0 \) and \(-c_i \), \( i \) closes some fraction of her contract position, and takes delivery of the remainder. Proposition 5 describes outcomes in the spot market, in a competitive-bidding benchmark, and in equilibrium.

**Proposition 5.** With physical delivery contracts, if spot traders behave competitively, as if residual supply in the spot market is perfectly elastic, spot traders’ bids are:

\[
    z_{Bi} (p; c_i, \psi) = -c_i + \kappa (p - \psi) \tag{41}
\]

There is no net trade of the spot good. Spot traders’ aggregate demand for the spot good is:

\[
    \sum_i c_i + z_{Bi} (p; c_i, \psi) = \sum_i \kappa (p - \psi) \tag{42}
\]

Hence, markets clear at price \( p = \psi \).

In equilibrium, spot traders’ optimal bids are:

\[
    z_{Bi} (p; c_i, \psi) = -\frac{n-2}{n-1} c_i + \frac{n-2}{n-1} \kappa (p - \psi) \tag{43}
\]

Spot traders’ net demand is identical to (7) of proposition 1:

\[
    q_{Bi} (p; c_i, \psi) = \frac{1}{n-1} c_i - \frac{n-2}{n-1} \kappa (p - \psi) \tag{44}
\]

The market clearing price is identical to (8) of proposition 1:

\[
    p - \psi = \frac{\sum_{i=1}^{n} c_i}{n(n-2) \kappa} \sim N \left( 0, \frac{\sigma_c^2}{n(n-2)^2 \kappa^2} \right) \tag{45}
\]

Effectively, proposition 5 shows that physical delivery and cash-settled contracts behave identically within the model. If spot traders behave competitively, ignoring their price impact, there is no net trade in the spot market: all spot traders set \( z_i = -c_i \), which corresponds to closing all their contract positions just before settlement. Spot prices are equal to the risk factor \( \psi \), so the hedger can perfectly hedge.
In equilibrium, (44) shows that the net quantity purchased by spot traders, taking into account their contract positions, is identical to what their bids would be in, (7), in the cash-settlement case. Thus, from (45), prices are identical to what they are under cash settlement, so all welfare implications for all agents are identical.

The intuition behind this equivalence, which is explained in detail in Kyle (2007), is that cash settlement and physical delivery contracts are exactly equivalent if traders have access to “market-on-expiration” orders: that is, if traders can buy or sell arbitrary quantities of the spot good at exactly the cash settlement price. This equivalence holds in the auction model of this paper. A trader who holds a physical delivery contract, and thus is entitled to \( c_i \) units of the spot good, can sell \( z_i = -c_i \) units, thus receiving no net units of of the spot good and a monetary payment of \( c_i p \), effectively converting her physical delivery contract position into a cash-settled position. Similarly, a cash-settled contract holder can submit an order for \( z_i = c_i \) units, converting her cash-settled contract into a physical delivery contract.\(^{22}\)

Proposition 5 shows that there are some differences in how manipulation occurs logistically, under cash-settled versus physical delivery contracts. For cash-settled contracts, a long contract holder manipulates by buying the spot good at settlement. For physical delivery contracts, comparing (41) and (43), contract holders manipulate by setting \( z_i \) somewhat smaller in absolute value than \( -c_i \): that is, a long contract holder closes less of her contract positions than she would in a competitive world, taking delivery on some fraction of her contracts. Her deliveries increase spot prices, increasing her payoffs on the contracts that she does close.

\(^{22}\)Kyle (2007) also discusses conditions under which this equivalence may not hold. For some price benchmarks, exact “market-on-expiration” orders are difficult or impossible – for example, when benchmarks are calculated as time-weighted average prices. One other form of manipulation, which is only possible with physical-delivery contracts, is that a long manipulator may aim to buy up enough of the underlying asset that it is essentially impossible for shorts to fulfill their delivery requirements. The manipulator then uses the threat of default to extract large payments from shorts to close out contracts. In cases like this, payments from shorts may depend not only on spot market conditions, but also their perceived costs of defaulting on contract obligations, so the model in this paper may not fully capture market outcomes in these cases.
6 Discussion

6.1 Implications for contract market regulation

This paper’s framework assists regulators in precisely defining contract market manipulation, and distinguishing it from other forms of strategic trading in imperfectly competitive markets. The results show how structural policy tools, such as position limits, can be used to reduce market participants’ incentives to manipulate.

Defining manipulation, and behavioral solutions. As subsection 2.2 discusses, contract market manipulation is illegal, but the law does not precisely define manipulation, and courts, regulators and academics have faced substantial difficulty in precisely defining manipulation, which is a major barrier to enforcement. The framework of this paper presents a potential definition of manipulation. When strategic spot market traders hold derivative contracts linked to spot markets, traders have incentives to modify their trading behavior, to increase the payoffs on their contract positions. This force is distinct from the phenomenon of “bid shading” in spot markets. If spot traders are barred from holding derivative contracts, they would shade bids, but not manipulate.

This definition illustrates patterns that regulators might look for, to demonstrate that market participants are engaging in illegal manipulation. Trading in spot markets, which could not be profitable on a standalone basis, may be profitable once taking contract positions into account. Moreover, traders may “over-hedge” in equilibrium, buying contract positions much larger than their factor exposures, deliberately building up exposure to factor risk, since they know they can manipulate contract settlement prices.

Some of the features in the model may still be difficult for regulators to bring to data. For example, while inventory or preference shocks are precisely defined within the model, these may be difficult to estimate in practice. Market participants who are accused of manipulation could thus always claim that their spot trades are driven by inventory or preference shocks, rather than the desire to manipulate. The paper’s results suggest that over-hedging is symptomatic of manipulation. Proving over-hedging requires estimating the size of market participants’ factor risk exposures, which may be difficult if market participants have positions in many different products with correlated values. Moreover, market participants may enter into positions to speculate rather than hedge risk, and a
market participant could always argue that a position which appears to be an over-hedge, with manipulative intent, was in fact simply a speculative bet on factor risk.

**Structural regulation.** Besides attempting to behaviorally regulate manipulation, regulators also use a variety of structural interventions in contract markets to limit manipulation risk. This paper’s results assist regulators in structural regulation, because it quantitatively illustrates the forces that affect manipulation risk. Manipulation depends on market liquidity, market competition, and the size of agents’ contract positions. There are thus a few ways regulators can intervene to lower manipulation risk.

Regulators could impose position limits on spot traders’ contract positions, especially close to contract settlement. Position limits are currently imposed in many derivative contract markets. However, the primary purpose of position limits seems to be preventing excessive price volatility. As a result, the CFTC currently applies position limits more harshly for pure hedgers than spot traders. The results of this paper suggest that, for alleviating manipulation risk, position limits should actually be more harsh for spot traders than pure hedgers.

Spot market liquidity and competitiveness also affect manipulation risk. Regulators should thus attempt to ensure that contracts are settled based on liquid and competitive markets. This suggests, for example, that a recent regulatory effort to move interest rate benchmarks to more liquid underlying markets could potentially alleviate manipulation risk. Moreover, since spot market structure affects spot traders’ manipulation incentives, regulators cannot determine manipulation incentives by focusing on derivative contract markets in isolation; instead, contract market regulators should monitor liquidity and concentration in spot markets, perhaps in collaboration with antitrust and spot market

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23 The CFTC’s stated purpose for speculative position limits is to “protect futures markets from excessive speculation that can cause unreasonable or unwarranted price fluctuations”; hence position limits do not exist solely to combat manipulation, although according to my theory they can be an effective tool for doing so.

24 Contract position limits do not apply for market participants who have bona fide commercial risks to hedge; see the CFTC’s website on Speculative Limits.

25 Regulators already regulate benchmark setting: principles for financial benchmarks have been released by the International Organization of Securities Commissions (IOSCO (2013)), the FCA began regulating a number of benchmarks, and then in 2018 EU law was revised to include benchmark regulation, along similar principles to the IOSCO report.

26 See, for example, Duffie and Stein (2015),
regulators, since spot market structure is an important input for determining contract market manipulation risk.

The paper also shows how to estimate the size of manipulation-induced market distortions, using market data which is often observed by regulators. These estimates could be used by regulators to identify contract markets which are vulnerable to manipulation, and to predict how policy interventions, such as tighter position limits, or changes to spot market structure, would influence manipulation risk. These estimates could also be used to estimate manipulation-induced damages, which may be useful in manipulation-related legal proceedings.

6.2 Mechanisms for benchmark setting

This paper uses uniform-price double auctions as a reduced-form model of price benchmarks. This is potentially a reasonable model for many, but not all, benchmarks. Some benchmarks, such as VIX and the LBMA gold price, are determined using actual auctions. Derivative contracts for many equity indices are also settled based on exchange opening or closing auction prices. Some benchmark-setting mechanisms may produce outcomes similar to uniform-price double auctions. The WM/Reuters FX fixing and the ISDAFIX interest rate swap benchmark (now the ICE swap rate) are set using exchange prices over short time periods; these outcomes may be reasonably well-modelled by uniform-price double auctions. Likewise, some benchmarks for commodities such as oil and gas are set using volume-weighted average prices in specific geographical locations, over relatively short time spans; if the underlying goods are relatively homogeneous, uniform-price auctions may be a reasonable model of outcomes.

Other benchmarks are less well approximated by auctions, and the results of this paper may apply less well in these settings. Some benchmarks are based on trades of underlying assets in markets with large search or transportation frictions. For example, the CME Feeder Cattle Index is based on US-wide cattle trade prices; the price of cattle

27 Understanding the Special Opening Quotation (SOQ)
28 WM/Reuters FX Benchmarks
29 ICE Swap Rate
30 Klemperer and Meyer (1989) shows that supply function competition between dealers produces outcomes equivalent to uniform-price double auctions.
traded in New York on any given day may differ substantially from the price of cattle traded in California. Other markets are organized as core-periphery networks, with central dealers trading with peripheral counterparties (Wang, 2017; Duffie and Wang, 2016). In these markets, agents’ manipulation incentives may depend on their physical location, or their position in the dealer network, in addition to their holding capacity for spot goods.

Some benchmarks are not based on prices of verifiable trades, but rely on market participants to self-report trades or potential trades. For example, LIBOR is based on banks’ announcements of their borrowing costs, and some natural gas benchmarks are based on market participants’ reports of their trades. In these settings, market participants can manipulate benchmarks simply by falsely reporting trades, so manipulation is potentially much easier than in the auction model studied in this paper.

7 Conclusion

The regulation of contract market manipulation is a contentious topic in both academic and policy circles. Illegal manipulation is essentially defined as trading with the intent to move prices. This definition is vague, and it is unclear that regulators and courts currently apply it in a way that can improve market quality and social welfare.

This paper develops a simple model of contract market manipulation. In the model, manipulation is a market failure, which can cause equilibrium outcomes to be Pareto dominated. Regulatory intervention, such as imposing position limits on spot traders, can alleviate this market failure, and in some cases can improving the welfare of all market participants. The model assists regulators in defining contract market manipulation precisely, and illustrates what behaviors are symptomatic of manipulation. The model also illustrates how market primitives affect the likelihood of manipulation, and how different policy tools can be used to limit manipulation risk. Together, these results of this paper can assist policymakers in their efforts to regulate manipulation in derivative contract markets.

31 ICE LIBOR
32 CFTC Press Release 5409-07
References


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Online Appendix

A Proof of proposition 1

Repeating (3), spot traders’ total wealth can be written as:

\[ W_{\text{spot}} (x_i, c_i, z_i, p, \psi) = \psi x_i - \mu \psi x_i - \mu \psi c_i + p c_i + p z_i - \frac{1}{2 \kappa} z_i^2 \]

\[ - \frac{p z_i}{2} \]

(46)

A.1 Spot market bidding

To solve the spot market auction, I adopt the standard solution concept of equilibrium in ex-post optimal bid curves. A bid curve is ex-post optimal if it is optimal for any realization of other agents’ bid curves which occurs in equilibrium. I further restrict attention to linear bid curves, and strategies which are symmetric across agents.

From the perspective of agent \( i \), the spot auction defines a residual supply curve, \( z_{RSi} (p) \), specifying the number of units of the underlying asset that \( i \) is able to trade at price \( p \). This is the negative of the sum of all other agents’ bid curves:

\[ z_{RSi} (p) = - \sum_{j \neq i} z_{Bj} (p; c_j; \psi) \]

(47)

In equilibrium, if bid curves are linear, residual supply functions will also be linear, with a fixed slope:

\[ z_{RSi} (p) = d (p - \psi) + \eta_i \]

(48)

Where the random intercept \( \eta_i \) depends on uncertainty in other traders’ bid curves, resulting from uncertainty in their contract positions \( c_i \). I solve the spot auction model using the standard Kyle (1989) trick: I assume agents can choose the quantity they want to purchase for every possible realization of \( \eta_i \), then show that these choices can be
implemented by an affine demand schedule.

Claim 3. In the spot market, given \( d \), agents’ optimal bid curves are:

\[
z_{Bi} (p; c_i, \psi) = \frac{\kappa}{d + \kappa} c_i - \frac{\kappa d}{d + \kappa} (p - \psi)
\]  

(49)

Proof. Spot trader \( i \)’s wealth is given by (46). Trader \( i \) chooses her bid curves after \( x_i \) and \( \psi \) are realized, so I analyze agents’ choices conditional on \( x_i \) and \( \psi \). Assume the agent faces a residual supply curve as described in (48), and rearrange to get the inverse residual supply function:

\[
p_{RS} (z_i; \eta_i, \psi) = \psi + \frac{z_i - \eta_i}{d}
\]  

(50)

Suppose \( i \) can condition her purchase decision on \( \eta_i \). We can write (46) as:

\[
W = \psi x_i - \psi \mu x + p_{RS} (z_i; \eta_i, \psi) c_i - \mu x c_i + z_i \psi - \frac{1}{2\kappa} z_i^2 - z_i p_{RS} (z_i; \eta_i, \psi)
\]  

(51)

All components of (51) are known to \( i \), so \( i \) simply chooses her purchase quantity \( z_i \) to maximize (51). Differentiate with respect to \( z_i \):

\[
p_{RS}' (z_i; \eta_i, \psi) c_i + \psi - z_i p_{RS}' (z_i; \eta_i, \psi) - p (z_i; \eta_i, \psi) - \frac{z_i}{\kappa} = 0
\]  

(52)

From (50), we have \( p_{RS}' (z_i; \eta_i) = \frac{1}{d} \), hence (52) becomes:

\[
\frac{c_i}{d} + \psi - \frac{z_i}{d} - p_{RS} (z_i; \eta_i, \psi) - \frac{z_i}{\kappa} = 0
\]  

(53)

Expression (53) implicitly defines the optimal choice of \( z_i \) given \( \eta_i \). Expression (53) defines an affine bid curve; solving for \( z_i \), we attain expression (49). Since (49) passes through exactly all pairs \( (z_i, p) \) which are \( i \)'s optimal choices for some realization of \( \eta_i \), \( i \) can do no better than submitting bid curve (49).

Now, note that from (47), the slope of residual supply \( d \) facing any given agent is equal to the sum of all \( n - 1 \) other agents’ bid slopes. Thus, in equilibrium, we must have:

\[
d = (n - 1) \frac{\kappa d}{d + \kappa}
\]
Solving for \(d\), we get:
\[
d = (n - 2) \kappa
\]  
(54)

Plugging this into (49), we get (7) of proposition 1. This implies that agents’ optimal bids are:
\[
z_{Bi} (p; c_i, \psi) = \frac{1}{n - 1} c_i - \frac{n - 2}{n - 1} \kappa (p - \psi)
\]
proving (7).

To get prices, sum bids and add to 0:
\[
\sum_{i=1}^{n} \frac{1}{n - 1} c_i - \frac{n - 2}{n - 1} \kappa (p - \psi) = 0
\]

Solving for \(p - \psi\), and using that agents’ contract positions are normally distributed with variance \(\sigma^2_c\), we get (8).

### A.2 Spot trader welfare conditional on \(x_i\)

Now, I analyze how much utility trader \(i\) achieves in expectation, if she has factor exposure \(x_i\). Suppose an agent with factor exposure \(x_i\) is bidding against a residual supply curve of the form (48).

Claim 4. Agent \(i\)’s expected utility given \(\alpha, \sigma^2_\psi, \kappa, x_i, c_i, \sigma^2_\eta, d\) is:
\[
\sqrt{\frac{d^2 + 2\kappa d}{\alpha \kappa \sigma^2_\eta + d^2 + 2\kappa d}} \left( - \exp \left( -\frac{\alpha}{2} \left( -\alpha \sigma^2_\psi (c_i + x_i)^2 - \frac{\alpha \sigma^2_\psi - \kappa}{\alpha \kappa \sigma^2_\eta + d^2 + 2\kappa c_i^2} \right) \right) \right)
\]  
(55)

**Proof.** To calculate expected utility over uncertainty in \(\eta_i\) and \(\psi\), we first write expected utility from the auction as a function of \(\eta_i\), fixing \(c_i\). Rearranging residual supply from (48), we have:
\[
p = \psi + \frac{z_i + \eta_i}{d}
\]  
(56)

Wealth is:
\[
W = \psi x_i - \mu_\psi x_i - \mu_\psi c_i + p c_i + z_i \psi - \frac{1}{2\kappa} z_i^2 - z_i p
\]
Plugging in (56) for prices and rearranging, we have:
\[ W = \psi x_i - \mu \psi x_i - \mu \psi c_i + \psi c_i + \frac{\eta_i c_i}{d} + \]
\[ z_i (\eta_i; c_i) c_i - \left( \frac{z_i (\eta_i; c_i)}{d} \right)^2 - \frac{\eta_i z_i (\eta_i; c_i)}{d} \]  
(57)

Now, to find an expression for \( z_i (\eta_i; c_i) \), we eliminate prices from expression (49) for optimal bid curves and expression (48) for residual supply, to get \( z_i \) as a function of \( \eta_i \):
\[ z_i (\eta_i; c_i) = \frac{\kappa}{d + 2\kappa} (c_i - \eta_i) \]  
(58)

Plugging (58) into expression (57) for wealth, and simplifying, we have that wealth is:
\[ \psi x_i - \mu \psi x_i - \mu \psi c_i + \psi c_i + \frac{\eta_i c_i}{d} + \frac{(c_i - \eta_i)^2 \kappa}{2d^2 + 4\kappa d} \]

Given our assumption of CARA utility, agents’ utility is:
\[ -\exp \left( -\alpha \left( \psi x_i - \mu \psi x_i - \mu \psi c_i + \psi c_i + \frac{\eta_i c_i}{d} + \frac{(c_i - \eta_i)^2 \kappa}{2d^2 + 4\kappa d} \right) \right) \]  
(59)

We first integrate (59) over uncertainty in \( \eta_i \), assuming that \( \eta_i \) is normally distributed with mean 0 and variance \( \sigma^2_{\eta_i} \), to get:
\[ -\sqrt{\frac{d^2 + 2\kappa d}{\alpha \kappa \sigma^2_{\eta_i} + d^2 + 2\kappa d}} \exp \left[ -\alpha \left( \psi x_i - \mu \psi x_i + \psi c_i - \mu \psi c_i \right) + \frac{\alpha}{2} \left( \frac{\alpha \sigma^2_{\eta_i} - \kappa}{\alpha \kappa \sigma^2_{\eta_i} + d^2 + 2\kappa d} \right) c_i^2 \right] \]  
(60)

This gives expected utility over uncertainty in \( \eta_i \), conditional on \( \psi \). Now, we integrate (60) over uncertainty in \( \psi \), which is normally distributed with mean \( \mu \psi \) and variance \( \sigma^2_{\psi} \), to get:
\[
\sqrt{\frac{d^2 + 2\kappa d}{\alpha \kappa \sigma_n^2 + d^2 + 2\kappa d}} \left( -\exp \left( -\frac{\alpha}{2} \left( -\alpha \sigma_{\psi}^2 (c_i + x_i)^2 - \frac{\alpha \sigma_n^2 - \kappa}{\alpha \kappa \sigma_n^2 + d^2 + 2\kappa d} c_i^2 \right) \right) \right)
\]

as desired. \qedhere

\section*{A.3 Optimal hedging}

Using claim 4, we can find spot traders’ optimal choice of \(c_i\). We conjecture that the equilibrium contract price is \(p_c = \mu_{\psi}\), and we prove this conjecture in Appendix A.7 below.

\textbf{Claim 5.} If:

\[1 + \frac{\alpha \sigma_n^2 - \kappa}{\alpha \sigma_{\psi}^2 (\alpha \kappa \sigma_n^2 + d^2 + 2\kappa d)} > 0 \quad (61)\]

then spot traders’ objective function is strictly concave in \(c_i\), and there is a unique optimal choice of \(c_i\), which satisfies:

\[c_i \left( 1 + \frac{\alpha \sigma_n^2 - \kappa}{\alpha \sigma_{\psi}^2 (d^2 + 2\kappa \alpha \kappa \sigma_n^2)} \right) = -x_i \quad (62)\]

\textbf{Proof.} Take conditional expected utility from (55). Since only the exponent depends on \(c\), and the function \(-\exp \left( -\frac{\alpha}{2} (x) \right)\) is increasing in \(x\), we choose \(c_i\) to maximize:

\[-\alpha \sigma_{\psi}^2 (c_i + x_i)^2 - \frac{\alpha \sigma_n^2 - \kappa}{\alpha \kappa \sigma_n^2 + d^2 + 2\kappa d} c_i^2 \quad (63)\]

Taking the second derivative, note that the problem is only concave if:

\[1 + \frac{\alpha \sigma_n^2 - \kappa}{\alpha \sigma_{\psi}^2 (d^2 + 2\kappa \alpha \kappa \sigma_n^2)} > 0 \]

proving (61). Assuming (61) holds, differentiate (63) with respect to \(c_i\) and rearrange to get (62). Claim 5, plugging in \(d = (n - 2) \kappa\) from (54), gives expressions (9) and (10) of
proposition \[ \square \]

A.4 Equilibrium $\sigma_{\eta}^2$

From (9) of proposition \[ \square \] traders’ optimal contract purchases are linear in their factor exposures $x_i$, and traders’ factor exposures $x_i$ are mean-0 normal, so traders’ equilibrium contract positions are also mean-0 normally distributed. Thus, the residual supply intercept term, $\eta$, is also mean-0 normally distributed. We can then solve the model by requiring agents’ optimal behavior given $\sigma_{\eta}^2$, the variance of the residual supply intercept $\eta$, to generate residual supply curves with variance $\sigma_{\eta}^2$.

Claim 6. For any $\alpha, \sigma_{\psi}^2, \kappa, \sigma_{x}^2, n$ there is a unique symmetric equilibrium value of $\sigma_{\eta}^2$, satisfying:

$$\sigma_{\eta}^2 = \frac{\sigma_{x}^2}{n-1} \left( 1 + \frac{\alpha \sigma_{\eta}^2 - \kappa}{\left( \alpha \sigma_{\psi}^2 \kappa \right) \left( (n^2 - 2n) \kappa + \alpha \sigma_{\eta}^2 \right)} \right)^{-2}$$

(64)

$$\sigma_{\eta}^2 > \frac{\kappa - \left( \alpha \sigma_{\psi}^2 \right) \left( n^2 - 2n \right) \kappa^2}{\alpha \left( 1 + \alpha \sigma_{\psi}^2 \kappa \right)}$$

(65)

Proof. Since (62) implies that contract positions $c_i$ are linear in exposures $x_i$, agents’ contract positions are also normally distributed, with mean 0 and variance:

$$\sigma_{c}^2 = \left( 1 + \frac{\alpha \sigma_{\eta}^2 - \kappa}{\left( \alpha \sigma_{\psi}^2 \kappa \right) \left( d^2 + 2dk + \alpha \kappa \sigma_{\eta}^2 \right)} \right)^{-2} \sigma_{x}^2$$

(66)

Now, note that residual supply facing $i$, summing over other agents’ bids using (7), is:

$$- \sum_{j \neq i} z_{Bi} (p; c_i) = - \sum_{j \neq i} \left( \frac{1}{n-1} c_i - \frac{n-2}{n-1} \kappa (p - \psi) \right)$$

hence, using representation (48) of residual supply, we have:

$$\eta = - \sum_{j \neq i} \frac{1}{n-1} c_i$$

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and all factor exposures $x_i$ are independent. Hence, the variance of the residual supply intercept, $\eta_i$, is:

$$\sigma^2_{\eta} = \frac{\sigma^2_c}{n-1}$$  \hspace{1cm} (67)

Combining this with (66), and plugging in $d = (n-2) \kappa$ from (54), we obtain (64). Note also that, using the definition of $t$ in (9), we can write (67) as:

$$\sigma^2_{\eta} = \frac{t^2 \sigma^2_x}{n-1}$$  \hspace{1cm} (68)

By claim 5 in order for (11) to solve agents’ optimal contract purchasing problem, the concavity condition (61) must also hold; plugging in $d = (n-2) \kappa$ to (61), we get:

$$1 + \frac{\alpha \sigma^2_{\eta} - \kappa}{\left(\alpha \sigma^2_{\eta} \kappa\right) \left((n^2 - 2n) \kappa + \alpha \sigma^2_{\eta}\right)} > 0$$  \hspace{1cm} (69)

Setting (69) to 0 and solving for $\sigma^2_{\eta}$, we get:

$$\kappa - \left(\alpha \sigma^2_{\eta}\right) \left(n^2 - 2n\right) \kappa^2 = \frac{\alpha \left(1 + \alpha \sigma^2_{\eta} \kappa\right)}{\alpha \left(1 + \alpha \sigma^2_{\eta} \kappa\right)}$$  \hspace{1cm} (70)

The LHS of (69) is increasing in $\sigma^2_{\eta}$, so (70) defines a lower bound for equilibrium values of $\sigma^2_{\eta}$. This is (65) of Claim 6. For notational convenience, we define the lower bound as:

$$M_{lower} \equiv \frac{\kappa - \left(\alpha \sigma^2_{\eta}\right) \left(n^2 - 2n\right) \kappa^2}{\alpha \left(1 + \alpha \sigma^2_{\eta} \kappa\right)}$$  \hspace{1cm} (71)

To show that there is a unique value of $\sigma^2_{\eta}$ satisfying (64) and (65), rearrange (64) to:

$$(n-1) \sigma^2_{\eta} = \left(1 + \frac{\alpha \sigma^2_{\eta} - \kappa}{\left(\alpha \sigma^2_{\eta} \kappa\right) \left((n^2 - 2n) \kappa + \alpha \sigma^2_{\eta}\right)}\right)^{-2} \sigma^2_x$$  \hspace{1cm} (72)

We want to study the behavior of (72), for $\sigma^2_{\eta}$ on the interval $(M_{lower}, \infty)$. This is true
because of monotonicity. The LHS of (72) is strictly increasing in $\sigma^2_\eta$, starting from 0 when $\sigma^2_\eta = 0$, and increasing unboundedly as $\sigma^2_\eta$ is large. On the interval $(M_{\text{lower}}, \infty)$, the RHS,

$$
\left(1 + \frac{\alpha \sigma^2_\eta - \kappa}{\alpha \sigma^2_\psi \kappa ((n^2 - 2n) \kappa + \alpha \sigma^2_\eta)}\right)^{-2} \sigma^2_\chi
$$

is strictly decreasing. As $\sigma^2_\eta$ approaches $M_{\text{lower}}$ from the right, the RHS is unbounded above. As $\sigma^2_\eta$ increases towards $\infty$, the RHS decreases towards the finite quantity:

$$
\left(1 + \frac{1}{\alpha \kappa \sigma^2_\psi}\right)^{-2} \sigma^2_\chi
$$

Hence, the LHS and RHS of (72) cross exactly once on the interval $(M_{\text{lower}}, \infty)$.

To visually depict this argument, appendix figure A.1 shows the LHS and RHS of (72) under different parameter settings. There are essentially two possibilities: the lower bound $M_{\text{lower}}$, defined in (70), can be positive or negative. The left plot shows a case where $M_{\text{lower}}$ is negative. For positive $\sigma^2_\eta$, the RHS begins from a constant value and decreases. The right plot shows a case where $M_{\text{lower}}$ is positive. The RHS is $\infty$ at $\sigma^2_\eta = M_{\text{lower}}$, and decreases towards a constant value. In either case, these curves must cross exactly once, and there is a unique equilibrium value of $\sigma^2_\eta$.

This proves claim 6 and thus (11) and (12) of proposition 1.

### A.5 Spot traders’ expected welfare over uncertainty in factor exposures

Plugging in $c_i = tx_i$, $d = (n - 2) \kappa$ to (55) of claim 4, we get expected utility conditional on $x_i$, for any linear contract purchasing rule:

$$
\sqrt{\frac{(n^2 - 2n) \kappa}{\alpha \sigma^2_\eta + (n^2 - 2n) \kappa}} \left(\exp \left(-\frac{\alpha}{2} \left(-\alpha \sigma^2_\psi (1 - t)^2 - \frac{\alpha \sigma^2_\eta - \kappa}{\alpha \sigma^2_\psi \kappa ((n^2 - 2n) \kappa + \alpha \sigma^2_\eta)} t^2 \right) x^2_i\right)\right)
$$

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Notes. Each panel shows the LHS (blue line) and RHS (red line) of (72), as we vary $\sigma_\eta^2$. In the left plot, we set $n = 3, \alpha = 1, \kappa = 1, \sigma_\psi^2 = 1, \sigma_x^2 = 1$. In the right plot, we set $n = 3, \alpha = 0.3, \kappa = 0.3, \sigma_\psi^2 = 1, \sigma_x^2 = 1$. The dotted vertical line in the right plot is the lower bound $M_{\text{lower}}$, defined in (71).

Integrating this against uncertainty in $x_i$, with mean 0 and variance $\sigma_x^2$, we get:

$$
- \sqrt{\frac{(n^2-2n)\kappa}{\alpha \sigma_\eta^2 + (n^2-2n)\kappa}} \sqrt{1 - \alpha \sigma_x^2 \left( \alpha \sigma_\psi^2 (1 - t)^2 + \left( \frac{\alpha \sigma_\eta^2 - \kappa}{\alpha \sigma_\psi^2} \right) \left( \frac{(n^2-2n)\kappa + \alpha \sigma_\eta^2}{(n^2-2n)\kappa + \alpha \sigma_\eta^2} \right) t^2 \right)}
$$

(73)

Substituting for $\sigma_\eta^2$ using (68), we get (13).

A.6 The hedger’s contract purchasing decisions and welfare

From (4), the hedger’s wealth is:

$$
W_{\text{hedger}} (x_i, c_i, p, \psi) = \psi x_i - \mu_\psi x_i - \mu_\psi c_i + pc_i
$$

Since the expectation of $p$, over uncertainty in $\psi$ and $x_i$, is $\mu_\psi$, the expectation of the hedger’s wealth is always equal to 0. To find the variance, adding and subtracting $\psi c_i$, 

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we can write this as:

\[ = \psi x_H - \mu_p x_H - \mu_p c_H + \psi c_H + (p - \psi) c_H \] (74)

Since the mean of the hedger’s wealth is independent of \( c_i \), the hedger simply purchases contracts to minimize the variance of wealth. Ignoring the constant terms, we can write (74) as:

\[ \psi x_H + (\psi + (p - \psi)) c_i \] (75)

The hedger chooses \( c_H \) to minimize the variance of (75), where \( p - \psi \) is independent of \( \psi \). This is a regression problem, and the solution is to choose \( c_i \) equal to negative the coefficient from regressing \( \psi x_H \) on \( p = \psi + (p - \psi) \):

\[ c_i = -\frac{\text{Cov} (\psi x_i, \psi + (p - \psi))}{\text{Var} (\psi + (p - \psi))} = -\frac{\sigma_{\psi x_H}^2}{\sigma_{\psi}^2 + \text{Var} (p - \psi)} \]

This proves (14). The residual wealth variance that the hedger is exposed to is \((1 - R^2)\) times the unhedged variance of wealth, \( x_H^2 \sigma_{\psi}^2 \):

\[ \text{Var} (\psi x_i + (\psi + (p - \psi)) c_i) = \left(1 - \frac{(\text{Cov} (x_i \psi, \psi + (p - \psi)))^2}{\text{Var} (\psi + (p - \psi)) \text{Var} (x_i \psi)} \right) x_H^2 \sigma_{\psi}^2 \]

\[ = \frac{\text{Var} (p - \psi)}{\sigma_{\psi}^2 + \text{Var} (p - \psi)} \sigma_{\psi}^2 x_H^2 \] (76)

Taking the expectation of (76) over \( x_H \), which has variance \( \sigma_{\psi}^2 \), we get (15).

### A.7 Equilibrium in the contract market

Finally, we show that it is an equilibrium for the market maker to set \( \bar{p}_c = \mu_p \). As in Kyle (1985), the market maker sets prices equal to the expectation of prices conditional on order flow. In any linear equilibrium, spot traders’ and the pure hedger’s contract positions \( c_i, c_H \) are linear in their factor exposures \( x_i, x_H \); since we have assumed spot traders’ factor exposures are infinitesimal relative to the hedger’s, spot traders’ contract positions are
also infinitesimal. Thus, total order flow for contracts is equal to the hedger’s order flow $c_H$, and the market maker sets price:

$$p_c = E[p | c_H]$$

We can write:

$$E[p | c_H] = E[\psi + (p - \psi) | c_H] = E[\psi | c_H] + E[p - \psi | c_H]$$

Now, we have assumed $c_H$ is independent of $\psi$, so $E[\psi | c_H] = \mu_\psi$. Moreover, since $x_i$ and $x_H$ are independent, $c_i$ and $c_H$ are independent, so from expression (8) for spot good prices, $c_H$ is independent of $p - \psi$. Hence, $E[p - \psi | c_H] = 0$. This shows that

$$E[p | c_H] = \mu_\psi$$

Thus, the market maker sets $p_c = \mu_\psi$ in the first stage. This completes the proof of proposition 1.

B Supplementary material for section 4

B.1 Comparative statics

Figure A.2 illustrates the effects of varying input parameters on equilibrium outcomes. When spot traders’ risk aversion $\alpha$ is low, traders are more willing to bear factor risk in order to attain manipulation profits, so the equilibrium and spot-trader-optimal values of $t$ increase, causing price variance to increase. When the spot good holding capacity $\kappa$ decreases, price variance increases. The welfare-maximizing value of $t$ for spot traders tends to be lower than the equilibrium $t$ when $\kappa$ is low, because the negative basis risk externalities from manipulation are larger.

Decreasing the variance of the risk factor, $\sigma^2_\psi$, makes spot traders more willing to buy large contract positions and manipulate, which increases $t$ in equilibrium and increases basis risk. Increasing agents’ factor exposure variance, $\sigma^2_x$, increases price variance, but
Notes. Comparative statics of the equilibrium and spot trader welfare-maximizing values of \( t \), spot traders’ equilibrium welfare gain (minus the competitive equilibrium value of -1), and equilibrium price variance, \( \text{Var} (p - \psi) \), as input parameters vary. The baseline values that parameters are varied around are \( n = 3, \alpha = 1, \kappa = 0.8, \sigma^2_{\psi} = 1.5, \sigma^2_x = 0.05 \).

actually decreases the equilibrium and socially optimal values of \( t \), as it becomes more costly for agents to deviate from full hedging. Larger factor exposures also imply that spot traders’ welfare losses are larger, and because negative basis risk externalities are larger, the spot-trader welfare maximizing \( t \) tends to fall below the equilibrium \( t \) when \( \sigma^2_x \) is large. Finally, increasing \( n \) causes all parameters to converge rapidly to their competitive values: the equilibrium \( t \) converges to 1, and price variance and net spot trader welfare losses from manipulation converge to 0.
B.2 Unboundedness of equilibrium $t$

Here, we show that the equilibrium value of $t$ is unbounded above. From (11) of Proposition 1, the equilibrium $\sigma^2_\eta$ must satisfy:

$$\begin{align*}
(n - 1) \sigma^2_\eta &= \left(1 + \frac{\alpha \sigma^2_\eta - \kappa}{(\alpha \sigma^2_\psi \kappa) \left((n^2 - 2n) \kappa + \alpha \sigma^2_\eta\right)}\right)^{-2} \sigma^2_x \tag{77}
\end{align*}$$

We can rearrange this to:

$$\frac{(n - 1) \sigma^2_\eta}{\sigma^2_x} = \left(1 + \frac{\alpha \sigma^2_\eta - \kappa}{(\alpha \sigma^2_\psi \kappa) \left((n^2 - 2n) \kappa + \alpha \sigma^2_\eta\right)}\right)^{-2} \tag{78}$$

Analogous to (77), the LHS of (78) is increasing in $\sigma^2_\eta$ and the RHS is decreasing, so the curves cross exactly once. From (10), the RHS of (78) is equal to $t^2$. Hence, to show that the equilibrium amount of overhedging can be unbounded above, we must show that, in equilibrium, the $y$-value of the point where the LHS and RHS of (78) cross is unbounded above.

To show this, consider any set of parameters such that the lower bound $M_{\text{lower}}$, defined in (71), is greater than 0. This is true when:

$$\left(\alpha \sigma^2_\psi\right) \left(n^2 - 2n\right) \kappa < 1$$

Intuitively, this requires that $n$, $\alpha$, $\sigma^2_\psi$, $\kappa$ are relatively small. The RHS of (78) then approaches $\infty$ as $\sigma^2_\eta$ approaches $M_{\text{lower}}$ from the right. As we decrease $\sigma^2_x$ towards 0, the $y$-value of the intercept increases unboundedly. This is depicted in Appendix Figure A.3 below. As we decrease $\sigma^2_\eta$, the LHS of (78) becomes steeper, and intersects the RHS (the purple line) at higher and higher $y$-values. The RHS is unbounded above, so the $y$-value of the intercept is also unbounded above.

Intuitively, what is happening in this example is that the positive lower bound $M_{\text{lower}}$ implies that there cannot be an equilibrium with low $\sigma^2_\eta$, in order for spot traders’ second-order condition to hold. But $\sigma^2_\eta$ depends on $t$ and $\sigma^2_x$, the variance of spot traders’ factor...
Notes. The dotted vertical line is the lower bound $M_{\text{lower}}$, defined in (71). The purple line is the RHS of (78), as $\sigma^2_\eta$ varies. The blue, red, yellow, and green lines show the LHS of (78), for different values of $\sigma^2_x$ in $\{0.1, 0.25, 0.5, 1\}$. For all lines, we set $n = 3$, $\alpha = 0.3$, $\kappa = 0.3$, $\sigma^2_\omega = 1$. 

risk exposures. As we decrease $\sigma^2_x$ towards 0, in order to sustain an equilibrium with a large equilibrium $\sigma^2_\eta$, spot traders must buy a large number of contracts for each unit of their factor risk exposures, so $t$ must increase unboundedly to keep $\sigma^2_\eta$ above the lower bound $M_{\text{lower}}$. An intuition is that spot traders must manipulate enough, and create enough basis risk, to discourage other spot traders from buying infinitely large contract positions.

**B.3 Convergence rate of equilibrium $t$**

From (11) and (10), we have:

$$\sigma^2_\eta = \frac{\sigma^2_x}{n-1} \left(1 + \frac{\alpha \sigma^2_\eta - \kappa}{\alpha \sigma^2_\omega \kappa ((n^2 - 2n) \kappa + \alpha \sigma^2_\eta)}\right)^{-2}$$  \hspace{1cm} (79)

and:

$$t = \left(1 + \frac{\alpha \sigma^2_\eta - \kappa}{\alpha \sigma^2_\omega \kappa ((n^2 - 2n) \kappa + \alpha \sigma^2_\eta)}\right)^{-2}$$  \hspace{1cm} (80)
The RHS of (80) is decreasing in $\sigma^2_{\eta}$. Hence an upper bound for the equilibrium $t$ comes from setting $\sigma^2_{\eta}$ to 0:

$$ t \leq \left( 1 - \frac{\kappa}{(\alpha \sigma^2_{\psi} \kappa) \left((n^2 - 2n) \kappa \right)} \right)^{-1} \quad (81) $$

Now, we can get an upper bound for $\sigma^2_{\eta}$ by plugging the upper bound on $t$, (81) into (79):

$$ \sigma^2_{\eta} \leq \frac{\sigma^2_{\chi}}{n-1} \left( 1 - \frac{\kappa}{(\alpha \sigma^2_{\psi} \kappa) \left((n^2 - 2n) \kappa \right)} \right)^{-2} \quad (82) $$

The RHS is decreasing in $n$, hence, we can set $n = 3$ in (82) to get:

$$ \sigma^2_{\eta} \leq M \equiv \frac{\sigma^2_{\chi}}{2} \left( 1 - \frac{\kappa}{\left(\alpha \sigma^2_{\psi} \kappa \right) \left(3 \kappa \right)} \right)^{-2} $$

where $M$ does not depend on $n$. Now, a lower bound for $t$ comes from plugging the upper bound $M$ into (80).

$$ t \geq \left( 1 - \frac{\alpha M - \kappa}{\left(\alpha \sigma^2_{\psi} \kappa \right) \left((n^2 - 2n) \kappa + \alpha M \right)} \right)^{-1} \quad (83) $$

Together, (81) and (83) bound the equilibrium value of $t$. Now, the function $\frac{1}{1-x}$ is differentiable at $x = 1$ with derivative equal to 1. Hence, (81) converges to 1 at the same rate that

$$ 1 - \frac{\kappa}{\left(\alpha \sigma^2_{\psi} \kappa \right) \left((n^2 - 2n) \kappa \right)} $$

converges to 1, which is $\frac{1}{n^2}$; (83) is analogous. Thus, the equilibrium value of $t - 1$ converges to 0 at rate $\frac{1}{n^2}$.  

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B.4 Heterogeneous beliefs

In this appendix, I show that contract purchases can also be generated by dispersion in agents’ beliefs about $\psi$. Suppose an agent believes the mean of $\psi$ is $\beta_{\psi i}$, and its variance is $\sigma^2_{\psi i}$. The agent can purchase contracts at a fixed price $\mu_{\psi}$, and has no factor exposure $x_i$. From (60), the agent’s utility as a function of $\psi$, over uncertainty in the spot market, is:

$$\sqrt{\frac{d^2 + 2\kappa d}{\alpha \kappa \sigma^2_{\psi} + d^2 + 2\kappa d}} \left(-\exp \left(-\alpha (\psi c_i - \mu_{\psi} c_i) - \frac{\alpha}{2} \left(\frac{\alpha \sigma^2_{\psi} - \kappa}{\alpha \kappa \sigma^2_{\psi} + d^2 + 2\kappa d} c_i^2 \right)\right)\right)$$

To calculate the agent’s expected utility (under her beliefs), we integrate assuming $\psi$ has mean $\beta_{\psi i}$ and variance $\sigma^2_{\psi i}$. This gives:

$$= \sqrt{\frac{d^2 + 2\kappa d}{\alpha \kappa \sigma^2_{\psi} + d^2 + 2\kappa d}} \left(-\exp \left(-\alpha \frac{\alpha \sigma^2_{\psi} c_i^2 + 2\beta_{\psi i} c_i - 2\mu_{\psi} c_i - \alpha \sigma^2_{\psi} - \kappa}{\alpha \kappa \sigma^2_{\psi} + d^2 + 2\kappa d c_i^2} \right)\right)$$

Only the exponent depends on $c_i$, so we maximize:

$$-\alpha \sigma^2_{\psi} c_i^2 + 2 (\beta_{\psi i} - \mu_{\psi}) c_i - \frac{\alpha \sigma^2_{\psi} - \kappa}{\alpha \kappa \sigma^2_{\psi} + d^2 + 2\kappa d} c_i^2$$

Differentiating and solving for $c_i$, we have:

$$c_i \left(1 + \frac{\alpha \sigma^2_{\psi} - \kappa}{(\alpha \sigma^2_{\psi}) (d^2 + 2\kappa d + \alpha \kappa \sigma^2_{\psi})} \right) = \frac{\beta_{\psi i} - \mu_{\psi}}{\alpha \sigma^2_{\psi}} \quad (84)$$

Comparing (84) to (9) and (10), a belief shock $\beta_{\psi i}$ is isomorphic to a factor exposure $x_i$ of size:

$$x_i = -\frac{\beta_{\psi i} - \mu_{\psi}}{\alpha \sigma^2_{\psi}}$$

in the sense that it generates the same contract purchasing decisions and spot market behavior. Hence, contract purchases can be motivated either by disagreement or risk-sharing. If we do not observe factor exposures directly, based only on agents’ contract purchases and spot market behavior, we cannot separately identify factor exposures from
heterogeneous beliefs, complicating welfare analysis for spot traders.

However, spot traders’ optimal behavior in spot markets only depends on the size of their contract positions and the structure of spot markets, not the motivations of spot traders for purchasing contracts. Hence, the results of section 5, analyzing the effects of manipulation on the hedger’s welfare, hold even if spot traders’ contract positions are driven by differences in beliefs.

**B.5 Price impact in the contract market, quadratic taxes and subsidies**

In this appendix, I show that, if spot traders’ contract purchases have price impact, we can still solve for linear optimal contract purchasing rules. Moreover, price impact is isomorphic to subsidies or taxes to spot traders which are quadratic in the size of traders’ contract positions. Thus, a regulator can implement any desired choice of hedging aggressiveness \( t \) in equilibrium using some quadratic subsidy or tax scheme.

Suppose that spot traders’ contract purchases move the spot price linearly: if a spot trader purchases \( c_i \) contracts, the price per contract is:

\[
\mu_\psi + \lambda c_i
\]

The total cost of buying \( c_i \) contracts is then:

\[
\mu_\psi c_i + \lambda c_i^2 \tag{85}
\]

Combining this cost with (3) and integrating, a spot trader’s conditional expected utility if she has factor exposure \( x_i \) and contract position \( c_i \) is:

\[
\frac{-1}{\sqrt{2\pi \sigma_\psi^2}} \int \sqrt{\frac{d^2 + 2\kappa d}{\alpha \kappa \sigma_\eta^2 + d^2 + 2\kappa d}} \exp \left[ -\alpha \left( \psi x_i - \mu_\psi x_i + \psi c_i - \mu_\psi c_i - \lambda c_i^2 \right) - \frac{\alpha^2}{2} \left( \frac{\alpha \sigma_\eta^2 - \kappa}{\alpha \kappa \sigma_\eta^2 + d^2 + 2\kappa d} \right) c_i \right] \exp \left( -\frac{(\psi - \mu_\psi)^2}{2\sigma_\psi^2} \right) d\psi
\]
\[
= \sqrt{\frac{d^2 + 2\kappa d}{\alpha \kappa \sigma_\eta^2 + d^2 + 2\kappa d}} \left( - \exp \left( -\frac{\alpha}{2} \left( -\alpha \sigma_\psi^2 (c_i + x_i)^2 - 2\lambda c_i^2 - \frac{\alpha \sigma_\eta^2 - \kappa}{\alpha \kappa \sigma_\eta^2 + d^2 + 2\kappa d} c_i^2 \right) \right) \right)
\]

Hence, agents choose \( c_i \) to maximize:

\[-\alpha \sigma_\psi^2 (c_i + x_i)^2 - 2\lambda c_i^2 - \frac{\alpha \sigma_\eta^2 - \kappa}{\alpha \kappa \sigma_\eta^2 + d^2 + 2\kappa d} c_i^2 \]

The optimal choice of \( c_i \) thus satisfies:

\[c_i = - \left( 1 + \frac{\alpha \sigma_\eta^2 - \kappa}{\alpha \sigma_\psi^2} \frac{\alpha \sigma_\eta^2 + \alpha \kappa \sigma_\eta^2}{(d^2 + 2\kappa)} + \frac{4\lambda}{\alpha \sigma_\psi^2} \right)^{-1} x_i \quad (86)\]

Substituting for \( d \) using the spot market equilibrium bid curves, we have:

\[c_i = - \left( 1 + \frac{\alpha \sigma_\eta^2 - \kappa}{\alpha \sigma_\psi^2} \frac{\alpha \sigma_\eta^2 + \alpha \kappa \sigma_\eta^2}{(n^2 - 2n) \kappa + \alpha \sigma_\eta^2} + \frac{4\lambda}{\alpha \sigma_\psi^2} \right)^{-1} x_i \quad (87)\]

Expression (87) shows that spot traders’ optimal contract positions are still linear in their factor exposures \( x_i \), with coefficient:

\[t \equiv \left( 1 + \frac{\alpha \sigma_\eta^2 - \kappa}{\alpha \sigma_\psi^2} \frac{\alpha \sigma_\eta^2 + \alpha \kappa \sigma_\eta^2}{(n^2 - 2n) \kappa + \alpha \sigma_\eta^2} + \frac{4\lambda}{\alpha \sigma_\psi^2} \right)^{-1} \quad (88)\]

Comparing (88) to (10) of proposition 1, price impact in the contract market decreases \( t \), causing spot traders to hedge less per unit of their factor exposures. Given this \( t \), the equilibrium \( \sigma_\eta^2 \) is the unique value that satisfies:

\[\sigma_\eta^2 = \frac{\sigma_x^2}{n - 1} \left( 1 + \frac{\alpha \sigma_\eta^2 - \kappa}{\alpha \sigma_\psi^2} \frac{\alpha \sigma_\eta^2 + \alpha \kappa \sigma_\eta^2}{((n^2 - 2n) \kappa + \alpha \sigma_\eta^2)} + \frac{4\lambda}{\alpha \sigma_\psi^2} \right)^{-2}\]

and outcomes in the spot market are described by (7) and (8). Qualitatively, price impact in contract markets simply causes spot traders to buy less contracts per unit of their factor.
expressions.

Expressions (86) and (88) also imply that regulators can implement any desired choice of $t$ as a unique equilibrium, by imposing quadratic taxes or subsidies on agents’ contract positions. To see this, suppose that a regulator can charge all spot traders some net amount $\tau c_i^2$ for buying $c_i$ contracts, where $\tau$ can be positive or negative. A spot trader’s total cost for buying $c_i$ contracts is thus:

$$\mu \psi c_i + \lambda c_i^2 + \tau c_i^2$$

Analogous to (88), spot traders’ optimal hedging decisions are linear, satisfying:

$$t \equiv \left(1 + \frac{\alpha \sigma^2 \eta - \kappa}{\left(\alpha \sigma^2 \psi\right) \left(d^2 + 2d\kappa + \alpha \kappa \sigma^2 \eta\right)} + \frac{4\lambda}{\alpha \sigma^2 \psi} + \frac{4\tau}{\alpha \sigma^2 \psi}\right)^{-1}$$  \hspace{1cm} (89)

Now, suppose the regulator wishes to implement some positive level of hedging aggressiveness, $t^* > 0$, in equilibrium. From (68) of appendix A.4, we have:

$$\sigma^2_n = \frac{(t^*)^2 \sigma^2}{n-1}$$

Hence, in order to implement $t^*$, we must choose $\tau$ such that:

$$t^* = \left(1 + \frac{\alpha \left(\frac{(t^*)^2 \sigma^2}{n-1}\right) - \kappa}{\left(\alpha \sigma^2 \psi\right) \left(d^2 + 2d\kappa + \alpha \kappa \left(\frac{(t^*)^2 \sigma^2}{n-1}\right)\right)} + \frac{4\lambda}{\alpha \sigma^2 \psi} + \frac{4\tau}{\alpha \sigma^2 \psi}\right)^{-1}$$

By changing $\tau$, the RHS can be varied from 0 to $\infty$, so for any $t^*$, and any values of other primitives, there is a unique value of $\tau$ which implements $t^*$ as an equilibrium outcome.

Throughout this appendix, we have taken price impact as exogenous. In a more realistic model, price impact would result endogenously from the contract market, in which spot traders and the hedger bid for contracts in a double auction. This is analytically difficult to solve, because spot traders’ order flow would be informative about their spot market trades, and thus contract settlement prices.
The $\lambda c_i^2$ term in (85) could also be used to model holding costs for contracts, which may arise from margin capital requirements or related factors. The results of this appendix then imply that quadratic holding costs, like price impact, decrease spot traders’ hedging aggressiveness in equilibrium. Once again, spot traders’ optimal behavior in spot markets depends only on the size of their contract positions and spot market structure. Thus, price impact in contract markets would not affect the results of section 5.

C Supplementary material for section 5

C.1 Proof of propositions 2 and 3

We prove proposition 3; proposition 2 is a special case. Claim 7 first characterizes agents’ best responses, given the slope of residual supply, and subsection C.1.2 proves proposition 3 using claim 7.

C.1.1 Best responses

Claim 7. If agent $i$ has inventory position $y_i$ and contract position $c_i$, and the slope of residual supply is $d_i$, then agent $i$’s unique ex-post optimal bid curve is:

$$z_{Bi} (p; y_i, c_i) = -\frac{d_i}{\kappa_i + d_i} y_i + \frac{\kappa_i}{\kappa_i + d_i} c_i - \frac{\kappa_i d_i}{\kappa_i + d_i} (p - \psi)$$ (90)

Proof. Analogously to claim 3, assume that residual supply takes the form:

$$z_{RSi} (p, \eta) = d_i (p - \psi) + \eta_i$$

We will optimize pointwise in the residual supply intercept $\eta_i$. Since we assumed all $c_i$ and $y_i$ have full support, in any linear bidding equilibrium, all $\eta_i$’s will also have full support. Define $p^* (\eta_i)$ as the optimal choice of $p$ for any given $\eta_i$, that is:

$$p^* (\eta_i) \equiv \arg\max_p W (z_{RSi} (p, \eta_i), p; y_i, c_i)$$

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arg max \( p \) \( \psi_{\text{RSi}} (p, \eta_i) - \frac{y^2_i}{2\kappa_i} - \frac{y_i z_{\text{RSi}} (p, \eta_i)}{\kappa_i} - \frac{z_{\text{RSi}} (p, \eta_i)^2}{2\kappa_i} + c_i p - z_{\text{RSi}} (p, \eta_i) p \)

Since \( z_{\text{RSi}} (p, \eta_i) \) is affine and increasing in \( p \), the objective function concave in \( p \), thus the first-order condition is necessary and sufficient for \( p^* (\eta_i) \) to be optimal. Differentiating with respect to \( p \) and setting to 0, and using that \( z'_{\text{RSi}} (p, \eta_i) = d_i \), we have:

\[-\frac{d_i}{\kappa_i} y_i - \frac{z_{\text{RS}} (p^* (\eta_i), \eta_i)}{\kappa_i} d_i + c_i - z_{\text{RS}} (p^* (\eta_i), \eta_i) - (p^* (\eta_i) - \pi_i) d_i = 0 \quad (91)\]

Hence, any pair \( (p^* (\eta_i), z_{\text{RS}} (p^* (\eta_i), \eta_i)) \) – that is, any point \( (p, z) \) which is the agent’s optimal choice for some \( \eta_i \) – satisfies (91). Hence, the unique bid curve which passes through the set of all ex-post optimal points is the curve implicitly defined by (91). Solving (91) for \( z_{\text{RS}} (p^* (\eta_i), \eta_i) \), we have (90). \(\Box\)

C.1.2 Equilibrium

This proof is based on Appendix A.4 of [Du and Zhu (2012)], with notational modifications to suit the context of this paper. We seek a vector of demand and residual supply slopes \( b_i \) which satisfy, for all \( i \):

\[ d_i = \sum_{j \neq i} b_i = B - b_i \quad (92) \]

\[ b_i = \frac{d_i \kappa_i}{\kappa_i + d_i} \quad (93) \]

Rearranging, we have:

\[ d_i = \frac{b_i \kappa_i}{\kappa_i - b_i} \quad (94) \]

Combining (92) and (94), we have:

\[ \sum_j b_j - b_i = \frac{b_i \kappa_i}{\kappa_i - b_i} \]

Defining \( B \equiv \sum_j b_j \), we have

\[ (\kappa_i - b_i) (B - b_i) = b_i \kappa_i \]

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This has two solutions. In order for $B > b_i$, we must pick:

$$b_i = \frac{2\kappa_i + B - \sqrt{B^2 + 4\kappa_i^2}}{2}$$  \hspace{1cm} (95)$$

This is (22) of proposition 2. $B$ must satisfy:

$$B = \sum_j b_j = \sum_{i=1}^n \frac{2\kappa_i + B - \sqrt{B^2 + 4\kappa_i^2}}{2}$$  \hspace{1cm} (96)$$

This is (23) in the main text. By multiplying the top and bottom of the RHS by $2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}$ and simplifying, this becomes:

$$B = \sum_{i=1}^n \frac{2\kappa_i B}{2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}}$$

Or,

$$B \left(-1 + \sum_{i=1}^n \frac{2\kappa_i}{2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}}\right) = 0$$  \hspace{1cm} (97)$$

Now, define

$$f(B) = -1 + \sum_{i=1}^n \frac{2\kappa_i}{2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}}$$

In order for $B$ to solve (97) when $B > 0$, we need $f(B) = 0$. Now, $f(0) > 0$, $f(B) \to -1$ as $B \to \infty$, and $f'(B) < 0$ for $B > 0$. Hence, $f(B) = 0$ at some unique $B$, hence there is a unique value of $B$ which solves (97), and thus there is a unique linear equilibrium for any demand slopes $\kappa_1 \ldots \kappa_n$.

Substituting $b_i = \frac{\kappa_i d_i}{\kappa_i d_i}$ into agents’ best-response bid curves from claim 7, we have agents’ equilibrium bids in expression (30). To find prices, sum agents’ demand curves and equate to 0:

$$\sum_{i=1}^n \left[-y_i \frac{b_i}{\kappa_i} + c_i \frac{b_i}{\sum_{j \neq i} b_j} - (p - \psi) b_i \right] = 0$$
Solving for $p$, we have (31).

C.2 Proof of claim 2

Proof. If contract positions are i.i.d., with variance $\sigma_c^2$, the variance of prices can be calculated by taking the variance of (21):

$$\text{Var}(p - \psi) = \frac{1}{B^2} \sum_{i=1}^{n} \left( \frac{b_i}{\sum_{j \neq i} b_j} \right)^2 \sigma_c^2$$

(98)

Now, using the upper and lower bounds we have upper and lower bounds for each $\frac{b_i}{\sum_{j \neq i} b_j}$ from (25) of claim 1, we have:

$$\frac{1}{B^2} \sum_{i=1}^{n} s_i^2 \sigma_c^2 \leq \frac{1}{B^2} \sum_{i=1}^{n} \left( \frac{b_i}{\sum_{j \neq i} b_j} \right)^2 \sigma_c^2 \leq \frac{1}{B^2} \sum_{i=1}^{n} \left( 1 + \frac{s_{\text{max}}}{1 - 2s_{\text{max}}} \right)^2 s_i^2 \sigma_c^2$$

$$\Rightarrow \frac{\sigma_c^2}{B^2} \sum_{i=1}^{n} s_i^2 \leq \frac{1}{B^2} \sum_{i=1}^{n} \left( \frac{b_i}{\sum_{j \neq i} b_j} \right)^2 \sigma_c^2 \leq \frac{\sigma_c^2}{B^2} \left( 1 + \frac{s_{\text{max}}}{1 - 2s_{\text{max}}} \right)^2 \sum_{i=1}^{n} s_i^2.$$

Using the definition of HHI in (26), we get (27).

C.3 Proof of corollary 1

Conjecture that:

$$b_i = \frac{n - 2}{n - 1} \kappa$$

This solves (22) and (23). Plugging this into (30) and (31), we get (32). To calculate price variance, set the sum of all agents’ bids to 0 and solve for price, to get:

$$p - \psi = \frac{1}{n \kappa} \sum_{i=1}^{n} \left[ -y_i + \frac{1}{n - 2} c_i \right]$$

Taking the variance, and using that we get (33).
C.4 Proof of proposition 4

For basis risk, we simply take the variance of expression (31). Prices can be written as:

\[ p - \psi = \frac{1}{B} (k'_y y + k'_c c) \]

Since we have assumed that contract positions and inventory shocks have mean 0, price variance can be written as:

\[
E \left[ (p - \psi)^2 \right] = E \left[ \frac{1}{B^2} (k'_y y + k'_c c) (k'_y y + k'_c c)' \right] = E \left[ \frac{1}{B^2} (k'_y y y' k_y + 2k'_y y c' k_c + k'_c c c' k_c) \right]
\]

(99)

Now, since we assumed each element of \( y \) and \( c \) has mean 0, we have:

\[
E [yy'] = \Sigma_{yy}, \quad E [yc'] = \Sigma_{yc}, \quad E [cc'] = \Sigma_{cc}
\]

(100)

Substituting these into (99), we get (35).

C.5 Manipulation rents

Define the expected manipulation rent as:

\[
E \left[ (p - \mu_\psi) \sum_{i=1}^{n} c_i \right]
\]

(101)

Expression (102) is the total expected transfer from spot traders to manipulators. I call this the expected manipulation rent, as it represents the rents that spot traders, as a group, extract from the hedger, as a result of spot traders’ ability to move prices in spot markets. This may be a useful number in regulatory proceedings, to quantify the total expected profits that spot traders as a group extract from the representative hedger. In the symmetric case, the expected manipulation rent is:
Claim 8. In the symmetric model, the expected manipulation rent is:

$$\tau \equiv \mathbb{E} \left( (p - \mu \psi) \sum_{i=1}^{n} c_i \right) = \frac{\sigma_c^2}{(n-2) \kappa} \quad (102)$$

Proof. Since we have assumed that $\psi$ is independent of factor exposures $x_i$, which are linearly related to contract positions $c_i$, we have:

$$\mathbb{E} \left[ \psi \sum_{i=1}^{n} c_i \right] = \mathbb{E} [\psi] \mathbb{E} \left[ \sum_{i=1}^{n} c_i \right] = \mathbb{E} [\mu \psi] \sum_{i=1}^{n} c_i$$

Hence,

$$\mathbb{E} \left[ (p - \mu \psi) \sum_{i=1}^{n} c_i \right] = \mathbb{E} \left[ (p - \psi) \sum_{i=1}^{n} c_i \right]$$

Now, plugging in for $(p - \psi)$ using (8) of proposition 1, this is:

$$\mathbb{E} \left( \left( \sum_{i=1}^{n} c_i \right) \frac{(\sum_{i=1}^{n} c_i)}{n (n - 2) \kappa} \right) \quad (103)$$

Since we have assumed agents’ factor exposures $x_i$ are independent, agents’ contract positions $c_i$ are also independent, so (103) becomes (102). □

Expression (102) behaves similarly to basis risk, (15). Both are higher when spot traders’ contract positions, $\sigma_c^2$, are large; when spot traders’ holding capacities $\kappa$ are low, so the price impact of spot trades is higher; and when $n$ is lower, so auctions are less competitive.

In the context of the model, only basis risk matters for the hedger’s welfare. This is because we have made the simplifying assumption that spot traders are infinitesimally small, so manipulation rents are infinitesimally small from the perspective of the hedger. Basis risk, on the other hand, affects the hedger and is not diffused away. The following claim shows that manipulation rents are also fairly simple to estimate in the general model, with similar data requirements to basis risk.
Claim 9. In the general model of proposition 3, manipulation rents are:

\[ \tau = \frac{1}{B} \left( -2k'_y \Sigma_{yc} \mathbf{1} + k'_c \Sigma_{cc} \mathbf{1} \right) \]  

(104)

where \( \mathbf{1} \) is the length-n unit vector.

Proof. We have:

\[
E \left[ (p - \psi) \left( \sum_{i=1}^{n} c^i \right) \right] = E \left[ \left( \frac{1}{B} (k'_y y^i + k'_c c^i) \right) (c^i \mathbf{1}) \right] = E \left[ \frac{1}{B} (k'_y y^i \mathbf{1} + k'_c c^i \mathbf{1}) \right]
\]

Substituting using (100), we get (104).

\[ \square \]

C.6 Proof of proposition 5

In the competitive case, spot traders’ wealth is described by (39). Spot traders’ marginal value of the spot good is thus:

\[ \psi - \frac{c_i}{\kappa} z_i - \frac{z_i}{\kappa} \]

If residual supply is perfectly elastic, spot traders bid to equate marginal values to the spot price:

\[ \psi - \frac{c_i}{\kappa} z_i - \frac{z_i}{\kappa} = p \]

Solving for \( z_i \), we get (41).

For equilibrium bidding, we largely follow appendix A.1. Traders face residual supply curve:

\[ z_{RSi} (p) = d (p - \psi) + \eta_i \]

Traders’ spot wealth is:

\[ (z_i + c_i) \psi - \frac{1}{2\kappa} (z_i + c_i)^2 - p_{RS} (z_i; \eta_i, \psi) z_i \]  

(105)

If we write (105) in terms of the net purchase quantity \( q_i \equiv z_i - c_i \), defined in (40), we get:

\[ q_i \psi - \frac{1}{2\kappa} q_i^2 - p_{RS} (z_i; \eta_i, \psi) (q_i - c_i) \]

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\[ p_{RS} (z_i; \eta_i, \psi) c_i + q_i \psi - \frac{q_i^2}{2\kappa} - p_{RS} (z_i; \eta_i, \psi) q_i \]  

Expression (106) is identical to (51), up to terms that do not depend on \( c_i, z_i, p_{RS} \). Thus, appendix A.1 implies that the ex-post equilibrium bid curves, expressed in terms of \( q_i \), are:

\[ q_{Bi} (p; c_i, \psi) = \frac{1}{n-1} c_i - \frac{n-2}{n-1} \kappa (p - \psi) \]  

this proves (44) of proposition 5. Substituting \( z_{Bi} \) for \( q_{Bi} \) in (107) using the fact that \( q_{Bi} = z_{Bi} + c_i \) from (40), we get (43) of proposition 5. To calculate equilibrium prices, note that markets clear when:

\[ \sum_i c_i + z_{Bi} (q; c_i, \psi) = \sum_i q_{Bi} (p; c_i, \psi) = 0 \]

Since the net-quantity bid curve (107) is identical to the bid curve (7) in proposition 1 of the baseline model, the market clearing price under physical delivery is identical to the market clearing price (8) of proposition 1 in the baseline model, proving (45).

### D Proof of claim 1

Fix \( \kappa_1 \ldots \kappa_N \), and let

\[ s_{max} = \frac{\max_i \kappa_i}{\sum_{i=1}^{n} \kappa_i} \]

In appendix D.1 below, I prove the following claim.

**Claim 10.** For any demand slopes \( \kappa_1 \ldots \kappa_N \), when \( s_{max} < \frac{1}{2} \), for all \( i \), we have:

\[ \left(1 - \frac{s_{max}}{1 - s_{max}}\right) \leq \frac{b_i}{\kappa_i} \leq 1 \]

Now, from claim 7 of appendix C.1, agents’ best-response bid curves are:

\[ z_{Bi} (p; y_i, c_i) = \frac{d_{i}}{\kappa_{i} + d_{i}} y_{i} + \frac{\kappa_{i}}{\kappa_{i} + d_{i}} c_{i} - \frac{\kappa_{i} d_{i}}{\kappa_{i} + d_{i}} (p - \pi) \]  

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We want to bound the difference between this and the approximation:

\[ z_{Bi}(p; y_i, c_i) \approx y_i + \frac{K_i}{\sum_{i=1}^n K_i} c_i - K_i (p - \pi) \quad (109) \]

Using claim 10, we can bound all three coefficients in the bid curve (108). Recall that \( b_i = \frac{K_i d_i}{K_i + d_i} \). Thus, we have:

\[
\left(1 - \frac{s_{\text{max}}}{1 - s_{\text{max}}}\right) K_i \leq \frac{K_i d_i}{K_i + d_i} \leq K_i
\]

\[ 1 - \frac{s_{\text{max}}}{1 - s_{\text{max}}} \leq \frac{d_i}{K_i + d_i} \leq 1 \]

Recall that \( d_i \equiv \sum_{j \neq i} b_i \). Thus,

\[
\frac{K_i}{K_i + d_i} = \frac{K_i}{K_i + \sum_{j \neq i} b_i}
\]

Using claim 10, we have:

\[
\frac{K_i}{K_i + \sum_{j \neq i} K_j} \geq \frac{K_i}{K_i + \sum_{j \neq i} b_i} \geq \frac{K_i}{K_i + \left(1 - \frac{s_{\text{max}}}{1 - s_{\text{max}}} - \sum_{j \neq i} K_j\right)} \quad (110)
\]

Now, note that:

\[
\frac{K_i}{K_i + \left(1 - \frac{s_{\text{max}}}{1 - s_{\text{max}}} - \sum_{j \neq i} K_j\right)} \geq \frac{K_i}{K_i + \left(1 - \frac{s_{\text{max}}}{1 - s_{\text{max}}}\right) \left(1 + \frac{s_{\text{max}}}{1 - 2s_{\text{max}}} \right) \sum_{j \neq i} K_j}
\]

Hence, (110) becomes:

\[ s_i \leq \frac{K_i}{K_i + \sum_{j \neq i} b_i} \leq \left(1 + \frac{s_{\text{max}}}{1 - 2s_{\text{max}}} \right) s_i \]

This proves claim 1, since from (108), \( \frac{K_i}{K_i + \sum_{j \neq i} b_i} \) is just another way to write \( \frac{b_i}{\sum_{j \neq i} b_j} \).
D.1 Proof of claim 10

From (22) in proposition 2, we have:

\[ b_i = \frac{B + 2\kappa_i - \sqrt{B^2 + 4\kappa_i^2}}{2} \]

This immediately implies that \( b_i \leq \kappa_i \), as \( b_i \) approaches \( \kappa_i \) from below as \( B \to \infty \). This proves the upper bound in claim 10.

\[ \frac{b_i}{\kappa_i} \leq 1 \]

Thus, we need only prove the lower bound:

\[ \left( 1 - \frac{s_{\text{max}}}{1 - s_{\text{max}}} \right) \leq \frac{b_i}{\kappa_i} \]  \hspace{1cm} (111)

We will proceed in two stages. Appendix D.1.1 proves claim 11 which states that, given \( s_{\text{max}} \) and \( \kappa_{\text{max}} \), we can construct an analytical lower bound for \( B \). Appendix D.1.2 then uses the lower bound for \( B \) to lower-bound the ratio \( \frac{b_i}{\kappa_i} \).

D.1.1 A lower bound for \( B \)

This subsection proves the following claim:

Claim 11. Fixing \( \kappa_{\text{max}}, s_{\text{max}} \), we have:

\[ B \geq \frac{2s_{\text{max}} - 1}{s_{\text{max}} (s_{\text{max}} - 1)} \kappa_{\text{max}} \]

Proof. From (23) in proposition 2 given \( \kappa_1 \ldots \kappa_N \), \( B \) satisfies:

\[ -1 + \sum_{i=1}^{n} \frac{2\kappa_i}{2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}} = 0 \]  \hspace{1cm} (112)
For any \( \kappa_1 \ldots \kappa_N \), define \( B(\kappa_1 \ldots \kappa_n) \) as the induced value of \( B \), that is:

\[
B(\kappa_1 \ldots \kappa_n) = \left\{ B : -1 + \sum_{i=1}^{n} \frac{2\kappa_i}{2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}} = 0 \right\}
\]

Now, without loss of generality, suppose agent \( i = 1 \) has the largest demand slope, so that:

\[ s_{\text{max}} = s_1 = \frac{\kappa_1}{\sum_{i=1}^{n} \kappa_i} \]

Alternatively, we can write this as:

\[ \sum_{i=2}^{n} \kappa_i = \left( \frac{1 - s_{\text{max}}}{s_{\text{max}}} \right) \kappa_1 \] (113)

Consider the following optimization problem:

\[
\begin{align*}
\min_{\kappa_2 \ldots \kappa_n} & \quad B(\kappa_1 \ldots \kappa_n) \\
\text{s.t.} & \quad \sum_{i=2}^{n} \kappa_i = \left( \frac{1 - s_{\text{max}}}{s_{\text{max}}} \right) \kappa_1, \quad 0 \leq \kappa_i \leq \kappa_1 \quad \forall i
\end{align*}
\] (114)

In words, problem (114) states that we choose \( \kappa_2 \ldots \kappa_n \) to minimize \( B(\kappa_2 \ldots \kappa_n) \), fixing \( \kappa_1 \) and \( s_{\text{max}} \) (which is equivalent to fixing the sum \( \sum_{i=2}^{n} \kappa_i \)). The minimal value of \( B \) from this problem is a lower bound for \( B(\kappa_1 \ldots \kappa_n) \) given \( \kappa_1 \) and \( s_{\text{max}} \). The following claim characterizes the solution to problem (114).

Claim 12. If \( \kappa_2 \ldots \kappa_n \) are an optimal solution to (114), then all but 1 element of \( \kappa_2 \ldots \kappa_n \) must be equal to either \( \kappa_1 \) or 0.

Proof. We use the the implicit function theorem to calculate the derivative \( \frac{dB}{d\kappa_i} \). Write expression (112) as:

\[
L = -1 + \frac{2\kappa_1}{2\kappa_1 + B + \sqrt{B^2 + 4\kappa_1^2}} + \sum_{i=2}^{n} \frac{2\kappa_i}{2\kappa_i + B + \sqrt{B^2 + 4\kappa_i^2}} = 0 \] (115)
Differentiate (115) with respect to $\kappa_i$, to get:

$$\frac{\partial L}{\partial \kappa_i} : \frac{2B \left( B + \sqrt{B^2 + 4\kappa_i^2} \right)}{\sqrt{B^2 + 4\kappa_i^2} \left( B + 2\kappa_i + \sqrt{B^2 + 4\kappa_i^2} \right)^2} \tag{116}$$

which is strictly positive. Differentiate (115) with respect to $B$ to get:

$$\frac{\partial L}{\partial B} : -\sum_{i=1}^{n} 2\kappa_i \left( 1 + \frac{B}{\sqrt{B^2 + 4\kappa_i^2}} \right) \left( B + 2\kappa_i + \sqrt{B^2 + 4\kappa_i^2} \right)^2 \tag{117}$$

This is strictly negative, so we have:

$$\frac{dB}{d\kappa_i} = -\frac{\partial L}{\partial \kappa_i} / \frac{\partial L}{\partial B} > 0 \tag{117}$$

Hence, $B$ is strictly increasing in $\kappa_i$. Differentiating (116) once again in $\kappa_i$, we have:

$$\frac{\partial^2 L}{\partial \kappa_i^2} = -\frac{2B}{(B^2 + 4\kappa_i^2)^2} < 0$$

Hence, $\frac{\partial L}{\partial \kappa_i}$ is strictly decreasing in $\kappa_i$. This implies that, if $\kappa_i > \kappa_j$, then $\frac{\partial L}{\partial \kappa_i} < \frac{\partial L}{\partial \kappa_j}$; from (117), this then implies that $\frac{dB}{d\kappa_i} < \frac{dB}{d\kappa_j}$. Now, suppose we have a vector of demand slopes $\kappa_2 \ldots \kappa_n$, such there exists indices $i, j$ such that $\kappa_i, \kappa_j$ are strictly between 0 and $\kappa_1$, with $\kappa_i \geq \kappa_j$. Then, since $\frac{dB}{d\kappa_i} \leq \frac{dB}{d\kappa_j}$, the following vector of demand slopes, for sufficiently small $\delta$, lowers the objective value in (114), while maintaining constraint satisfaction:

$$\left( \kappa_2 \ldots \kappa_i + \delta, \ldots \kappa_j - \delta, \ldots \kappa_n \right)$$

In words, this says that, since $\frac{dB}{d\kappa_i}$ is lower for higher $\kappa_i$, we can always lower the objective by increasing high values of $\kappa_i$ and decreasing lower values to keep the sum constant. Thus, if $\kappa_2 \ldots \kappa_n$ are optimal for problem (114), there cannot exist two indices $i, j$ such that $\kappa_i$ and $\kappa_j$ are both strictly between 0 and $\kappa_1$; thus, the solution to (114) must have all
but one element of $\kappa_2 \ldots \kappa_N$ equal to either $\kappa_1$ or 0, proving claim 12.

The bound from claim 12 is not yet useful, as the optimal value from problem (114) does not admit a simple analytical expression. However, the limit of a sequence of relaxations of problem (114) does yield an analytically tractable solution. I construct a sequence of relaxations of problem (114), parametrized by the integer $h \in \{1, 2, \ldots \infty\}$. For any $h$, let $\kappa_h(x)$ be a function defined on the interval $[2, n + 1]$, which is piecewise constant on intervals of length $\frac{1}{h}$. Define $B(\kappa_h(x))$ as:

$$\begin{align*}
B(\kappa_h(x)) &= \left\{ B : -1 + \frac{2\kappa_1}{2\kappa_1 + B + \sqrt{B^2 + 4\kappa_1^2}} + \int_2^n \frac{2\kappa_h(x)}{2\kappa_h(x) + B + \sqrt{B^2 + 4(\kappa_h(x))^2}} \, dx = 0 \right\}
\end{align*}$$

(118)

Define the $h$th minimization problem as:

$$\begin{align*}
\min_{\kappa_h(x)} B(\kappa_h(x)) \\
\text{s.t. } \int_2^n \kappa_h(x) = K(1 - s_{\text{max}}), \ 0 \leq \kappa_i \leq \kappa_1 \ \forall i
\end{align*}$$

(119)

Effectively, (118) splits each $\kappa_2 \ldots \kappa_n$ value in the original problem (114) into $h$ components, which may have different values of $\kappa_i$. To see that problem (119) is a strict relaxation of the original optimization problem (114), note that if we constrain $\kappa_h(x)$ to be constant on the intervals

$$[2, 3), [3, 4), \ldots [n, n + 1)$$

then both the objective and the constraints in problem (119) reduced to the original problem (114). Thus, any value attainable in the original optimization problem is attainable in the relaxed optimization problem, so the optimal value from problem (119), for any $h$, is a lower bound for the original problem.

The integral in the relaxed problem (118) is simply a weighted sum over a finite number of values of $\kappa_h(x)$; thus, as in the original problem, the objective function for the relaxed problem is concave in the value of $\kappa_h(x)$ on any interval. Thus, claim 12 characterizing the solution to the original problem applies to the relaxed problem for any
h: at any optimal solution, \( \kappa_h(x) \) must be equal to either 0 or \( \kappa_1 \) on all intervals but one. Taking the limit as \( h \to \infty \), the constraint

\[
\int_2^n \kappa_h(x) = K (1 - s_{\text{max}})
\]

implies that \( \kappa_h(x) \) must be equal to \( \kappa_1 \) on a set with measure arbitrarily close to

\[
\frac{1}{s_{\text{max}}} - 1
\]

and, \( \kappa_h(x) = 0 \) otherwise, except for an interval which has measure arbitrarily close to 0. Hence, in the limit as \( h \to \infty \), any choice of \( \kappa_h(x) \) which minimizes \( B \) has:

\[
\int_2^n \frac{2\kappa_h(x)}{2\kappa_h(x) + B + \sqrt{B^2 + 4(\kappa_h(x))^2}} \, dx \to \left( \frac{1}{s_{\text{max}}} - 1 \right) \frac{2\kappa_1}{2\kappa_1 + B + \sqrt{B^2 + 4\kappa_1^2}}
\]

Using (118), in the limit as \( h \to \infty \), the minimized value of \( B \) thus satisfies:

\[
-1 + \frac{2\kappa_1}{2\kappa_1 + B + \sqrt{B^2 + 4\kappa_1^2}} + \left( \frac{1}{s_{\text{max}}} - 1 \right) \frac{2\kappa_1}{2\kappa_1 + B + \sqrt{B^2 + 4\kappa_1^2}} = 0
\]

\[
\implies -1 + \frac{1}{s_{\text{max}}} \frac{2\kappa_1}{2\kappa_1 + B + \sqrt{B^2 + 4\kappa_1^2}} = 0
\]

This can be analytically solved for \( B \), to get:

\[
B = \frac{2s_{\text{max}} - 1}{s_{\text{max}} (s_{\text{max}} - 1)} \kappa_1
\]

This is thus a lower bound for the optimal value to the original optimization problem (114), hence, this is a lower bound for \( B \) given \( \kappa_1 \) and \( s_{\text{max}} \). It is a nontrivial lower bound whenever \( s_{\text{max}} < \frac{1}{2} \); otherwise, it is nonpositive. Since we assumed \( \kappa_1 = \kappa_{\text{max}} \), this proves claim [11]. This lower bound is tight whenever \( s_{\text{max}} = \frac{1}{n} \) for some integer value of \( n \), as it is exactly the equilibrium value of \( B \) when there are \( n \) agents with identical demand slopes, \( \kappa_i = \kappa \).
D.1.2 Bounding $b_i$

Now, we use claim 11 to prove the lower bound, (111), of claim 10. We have:

$$b_i = \frac{2\kappa_i + B - \sqrt{B^2 + 4\kappa_i^2}}{2}$$  \hspace{1cm} (120)

This shows that $b_i$ is increasing in $B$. Thus, we can find a lower bound for the ratio $\frac{b_1}{\kappa_1}$ for the largest agent by plugging in the lower bound for $B$ from claim 11 and solving for $b_i$. This gives:

$$\frac{b_1}{\kappa_1} \geq \left(1 - \frac{s_{\text{max}}}{1 - s_{\text{max}}} \right) \hspace{1cm} (121)$$

To bound the ratio $\frac{b_i}{\kappa_i}$ for all other $i$, I show that the ratio $\frac{b_i}{\kappa_i}$ is decreasing in $\kappa_i$, for $B$ fixed. To see this, differentiate (120), fixing $B$, to get:

$$\frac{db_i}{d\kappa_i} = \frac{1 - 2\kappa_i}{\sqrt{B^2 + 4\kappa_i^2}}$$

This is decreasing in $\kappa_i$, so $b_i$ is a concave function of $\kappa_i$ fixing $B$, and $b_i = 0$ when $\kappa_i = 0$. Now,

$$\frac{d}{dx} \frac{b_i(\kappa_i)}{\kappa_i} = \frac{\kappa_i b_i' (\kappa_i) - b_i (\kappa_i)}{\kappa_i^2}$$

This is negative if the numerator is negative; using that $b_i(0) = 0$, write the numerator as:

$$\kappa_i b_i' (\kappa_i) - b_i (\kappa_i) = \int_0^{\kappa_i} b_i' (\kappa) \, d\kappa$$

Since $b_i' (\kappa_i)$ is a decreasing function, $b_i' (\kappa_i) \geq b_i' (\kappa)$ for all $\kappa \leq \kappa_i$, hence the integrand is everywhere nonpositive; hence $\frac{d}{dx} \frac{b_i(\kappa_i)}{\kappa_i} \leq 0$ for all $\kappa_i > 0$. Thus, $\frac{b_i}{\kappa_i}$ is a decreasing function of $\kappa_i$ fixing $b_i$. This gives us:

$$\frac{b_i}{\kappa_i} \geq \frac{b_1}{\kappa_1} \geq \left(1 - \frac{s_{\text{max}}}{1 - s_{\text{max}}} \right)$$  

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which is a nontrivial lower bound when $s_{\text{max}} < \frac{1}{2}$. This proves the lower bound (111),
and thus proves claim 10.