Implementability, Walrasian Equilibria, and Efficient Matchings*

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Abstract

In general screening problems, implementable allocation rules correspond exactly to Walrasian equilibria of an economy in which types are consumers with quasilinear utility and unit demand. Due to the welfare theorems, an allocation rule is implementable if and only if it induces an efficient matching between types and goods.

JEL Codes: D50, D82, D86.

1 Introduction

In a *screening problem*, a principal who faces uncertainty over an agent's type designs a *contract* – an allocation rule mapping types to goods, and a transfer function mapping types to monetary payments – to maximize some objective function, such as profit or social welfare. Since the agent's type is private information, the contract must be *incentive compatible*. A classic issue in the theory of screening is the problem of *implementability* – given an allocation rule, when do there exist transfer functions under which truthful reporting is incentive compatible?

In this paper, we demonstrate an analogy between implementable allocation rules, Walrasian equilibria, and efficient matchings. Given a candidate allocation rule, we con-

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struct a quasilinear unit-demand economy in which each type of the agent is represented by a consumer with unit demand, and each assigned good corresponds to an indivisible commodity. By a version of the Taxation Principle, implementable allocations and their associated transfers correspond to Walrasian equilibrium allocations and prices of these economies. By the Welfare Theorems for quasilinear economies, Walrasian equilibrium allocations constitute efficient matchings between consumers and commodities.

The classic reference on implementability in general screening problems is Rochet (1987), who defines a "cyclic monotonicity" condition which is necessary and sufficient for an allocation rule to be implementable. Recently, a number of authors (Rahman, 2011; Hartline et al., 2015; Shao, 2014) have studied connections between implementability and efficient matchings. However, to our knowledge, the connection we draw between implementable allocation rules and Walrasian equilibria of quasilinear economies is new to the literature.

We emphasize three advantages of our approach. Firstly, the analogy to Walrasian equilibria provides an economically intuitive proof of the connection between implementable allocation rules and efficient matchings, through the Welfare Theorems for quasilinear economies. Secondly, as we show in Corollary 1, our analogy allows us to draw parallels between classical results about Walrasian equilibrium prices and recent results (Carbajal and Ely, 2013; Heydenreich et al., 2009; Kos and Messner, 2013) about the structure of incentive compatible transfers in screening problems without revenue equivalence. Finally, in Corollary 2, we demonstrate a novel result: since Walrasian equilibria exist in economies with arbitrary endowments, any allocation rule has at least one permutation which is implementable, and the implementable permutation is generically unique.

2 Model

2.1 The Screening Problem

There is a single agent with type $\theta \in \Theta \equiv \{\theta_1, \theta_2, \dots, \theta_n\}$, unobserved by the principal. The assumption that Θ is finite is mainly for clarity of exposition – we extend our main result to an arbitrary type space in Appendix C. $\mathcal{X}$ describes the set of decisions that the principal can take to affect the utility of the agent. In addition, the principal can impose monetary payments $t \in \mathbb{R}$. The agent has utility

$$v(\theta, x) - t,$$

where $v : \Theta \times \mathcal{X} \to \mathbb{R}$ is an arbitrary function. There is an *outside option* $\emptyset$ which is always available to the agent, with utility normalized to $v(\theta, \emptyset) = 0$, for all θ.

The principal chooses an *allocation rule* $\mathbf{x} : \Theta \to \mathcal{X}$.

Definition 1. Allocation rule $\mathbf{x}$ is *implementable* if there exists a *transfer function* $\mathbf{t} : \Theta \to \mathbb{R}$ such that the incentive compatibility and individual rationality constraints hold for each type $\theta \in \Theta$:

$$v(\theta, \mathbf{x}(\theta)) - \mathbf{t}(\theta) \geqslant v(\theta, \mathbf{x}(\hat{\theta})) - \mathbf{t}(\hat{\theta}), \quad \forall \hat{\theta} \in \Theta, \tag{IC}$$

$$v(\theta, \mathbf{x}(\theta)) - \mathbf{t}(\theta) \geqslant 0. \tag{IR}$$

The elements of $\mathcal{X}$ can be arbitrary objects, for example, state-contingent contracts, lotteries, or bundles. For concreteness, we will refer to elements of $\mathcal{X}$ as *goods*. For a given allocation rule $\mathbf{x}$, we denote by $C_{\mathbf{x}} = \{\mathbf{x}(\theta_1), \ldots, \mathbf{x}(\theta_n)\}$ the *collection* of n goods assigned under $\mathbf{x}$. As allocation rules may assign the same good to multiple types, we consider duplicates as distinct elements. We will use ω to denote a generic element of $C_{\mathbf{x}}$.

2.2 The Quasilinear Unit-Demand Economy

For a given allocation rule $\mathbf{x}$, define the economy $E_{\mathbf{x}} = \{C_{\mathbf{x}}; U_1, ..., U_n\}$, with the n commodities $C_{\mathbf{x}} = \{\mathbf{x}(\theta_1), \ldots, \mathbf{x}(\theta_n)\}$ and n consumers. Each consumer i can hold at most one indivisible commodity $\omega \in C_{\mathbf{x}}$, and has utility

$$U_i(\omega, t) = v(\theta_i, \omega) - t,$$

where t denotes a net monetary transfer from i. The value of not holding any commodity is zero. Consumer i is endowed with $\mathbf{x}(\theta_i)$ and a large amount of money.

A *feasible allocation* in $E_{\mathbf{x}}$ is a partition $\mathbf{y} = \{y_0, y_1, ..., y_n\}$ of $C_{\mathbf{x}}$, where y_i is the set of goods held by consumer i, with cardinality at most one, and y_0 is the set of goods not held by any consumer. A feasible allocation $\mathbf{y}$ *clears the market* if $y_0 = \emptyset$. Because there are n consumers and n commodities, market clearing requires each consumer to hold a commodity, so feasible market-clearing allocations can be viewed as bijective functions, or matchings, between consumers and commodities. For a feasible market-clearing allocation $\mathbf{y}$, we use y_i to mean the commodity held by consumer i.

A *price function* $\mathbf{p}$ is a mapping from $C_{\mathbf{x}}$ to $\mathbb{R}$, where $\mathbf{p}(\omega)$ represents the price of good ω. Since $|C_{\mathbf{x}}| = n$, we can think of $\mathbf{p}$ as a vector in $\mathbb{R}^n$.

Definition 2. A *Walrasian equilibrium* is a feasible market-clearing allocation $\mathbf{y}$ and a price function $\mathbf{p}$ such that each consumer i is optimizing given prices $\mathbf{p}$:

$$v(\theta_i, y_i) - \mathbf{p}(y_i) \geqslant v(\theta_i, \omega) - \mathbf{p}(\omega), \quad \forall \omega \in C_{\mathbf{x}}, \tag{WE1}$$

$$v(\theta_i, y_i) - \mathbf{p}(y_i) \geqslant 0. \tag{WE2}$$

Any allocation rule $\mathbf{x}$ defines a particular feasible market-clearing allocation $\mathbf{y}^{\mathbf{x}}$ in $E_{\mathbf{x}}$ by $y_i^{\mathbf{x}} = \mathbf{x}(\theta_i)$, for all i. Again, $\mathbf{y}^{\mathbf{x}}$ can also be thought of as a particular matching between the type space Θ and the collection of goods $C_{\mathbf{x}}$.

2.3 Matchings

We will consider all possible matchings between Θ and the $C_{\mathbf{x}}$. Let $\mathcal{M}(\Theta, C_{\mathbf{x}}) = \{\mathbf{y} : \Theta \to C_{\mathbf{x}} : \mathbf{y} \text{ is bijective}\}$. We call $\mathbf{y}^{\mathbf{x}}$ an *efficient matching* if it achieves higher total utility than any other matching in $\mathcal{M}(\Theta, C_{\mathbf{x}})$; that is, $\mathbf{y}^{\mathbf{x}}$ is an efficient matching if

$$\mathbf{y}^{\mathbf{x}} \in \arg\max_{\mathbf{y} \in \mathcal{M}(\Theta, C_{\mathbf{x}})} \sum_{i=1}^{n} v(\theta_i, \mathbf{y}(\theta_i)).$$

3 Results

Theorem 1. *The following statements are equivalent:*

1. *The allocation rule* $\mathbf{x}$ *is implementable.*
2. $\mathbf{x}$ *defines a Walrasian equilibrium allocation in* $E_{\mathbf{x}}$*.*
3. $\mathbf{y}^x$ *is an efficient matching.*

We prove the theorem by showing that (1) and (2) are equivalent, and then that (2) and (3) are equivalent.

Lemma 1. *[Taxation Principle]* The allocation rule $\mathbf{x}$ is implementable if and only if $\mathbf{x}$ defines a Walrasian equilibrium allocation in $E_{\mathbf{x}}$.

Proof. The conditions (IC) and (IR) are identical to conditions (WE1) and (WE2). If the allocation rule $\mathbf{x}$ is implemented by transfers $\mathbf{t}$, then $\mathbf{y}^x$ is a Walrasian equilibrium allocation in $E_{\mathbf{x}}$ under prices $\mathbf{p}\left(y_i^x\right) = \mathbf{t}(\theta_i)$, for all i. Conversely, if $\mathbf{y}^x$ is a Walrasian equilibrium allocation under prices $\mathbf{p}$, then transfers $\mathbf{t}(\theta_i) = \mathbf{p}\left(y_i^x\right)$, for all i, implement the allocation rule $\mathbf{x}$. □

Lemma 2. *[First and Second Welfare Theorems]* $\mathbf{x}$ defines a Walrasian equilibrium allocation in $E_{\mathbf{x}}$ if and only if $\mathbf{y}^x$ is an efficient matching.

Proof. See Appendix A. □

3.1 Corollaries

The triple equivalence of Theorem 1 allows us to draw connections between results known separately in each of the contexts. The following corollary characterizes the structure of the set of incentive compatible contracts for a given collection of goods. These results are not novel, as they are the subject of a recent series of papers (Carbajal and Ely, 2013; Heydenreich et al., 2009; Kos and Messner, 2013) studying the structure of incentive compatible transfer functions in settings where revenue equivalence may not hold. Our proof highlights the connection between this literature and classical results (Shapley and Shubik, 1971) about the structure of Walrasian equilibrium prices in quasilinear economies.

Given a contract (x, t), we define $\tau : C_x \to \mathbb{R}$ by $\tau(\omega) = t\left(x^{-1}(\omega)\right)$. Any contract (x, t) can be equivalently described by the pair (x, τ). Let $\Lambda(C)$ be the set of all incentive compatible contracts (x, τ) with $C_x = C$.

Corollary 1. *For any n-element collection C, $\Lambda(C)$ is a product set $\mathbf{X} \times \mathbf{T}$, where $\mathbf{X}$ is the set of efficient matchings between Θ and C, and $\mathbf{T}$ is a convex complete sublattice of $\mathbb{R}^n$ that is bounded from above.*

Proof. See Appendix B. □

We also demonstrate the following corollary, which to our knowledge has not appeared in the literature.

Corollary 2. *For any n-element collection C:*

1. *There exists an implementable allocation rule x with $C_x = C$.*
2. *For generic values of $v(\cdot, \cdot)$, there is a unique implementable allocation rule x with $C_x = C$.*

Equivalently, for any allocation rule x, there exists a permutation of x which is implementable, and the implementable permutation is generically unique.

Proof. Efficient matchings exist between Θ and any C. For generic choices of $v(\cdot, \cdot)$, the efficient matching is unique. The Corollary then follows from Theorem 1. □

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A Proof of Lemma 2

Proof. As Shapley and Shubik (1971) discuss, the welfare theorems in quasilinear unit-demand economies are equivalent to the duality theory of a linear program known as the *assignment problem*. We include the standard duality proof here for completeness.

Only if, First Welfare Theorem: For sake of contradiction, suppose $\mathbf{x}$ defines a Walrasian equilibrium under prices $\mathbf{p}$ and $\mathbf{y}^{\mathbf{x}}$ is not an efficient matching. Then there exists a matching $\mathbf{y}$ with $C_{\mathbf{y}} = C_{\mathbf{x}}$ such that:

$$\sum_{i=1}^{n} v(\theta_i, \mathbf{y}(\theta_i)) > \sum_{i=1}^{n} v(\theta_i, \mathbf{x}(\theta_i))$$

$$\implies \sum_{i=1}^{n} (v(\theta_i, \mathbf{y}(\theta_i)) - \mathbf{p}(\mathbf{y}(\theta_i))) > \sum_{i=1}^{n} (v(\theta_i, \mathbf{x}(\theta_i)) - \mathbf{p}(\mathbf{x}(\theta_i)))$$

$$\implies \exists\, i,\ v(\theta_i, \mathbf{y}(\theta_i)) - \mathbf{p}(\mathbf{y}(\theta_i)) > v(\theta_i, \mathbf{x}(\theta_i)) - \mathbf{p}(\mathbf{x}(\theta_i))$$

Hence, some consumer i prefers to purchase good $\mathbf{y}(\theta_i)$ over good $\mathbf{x}(\theta_i)$ under prices $\mathbf{p}$, contradicting (WE1).

If, Second Welfare Theorem: Suppose that $\mathbf{y}^{\mathbf{x}}$ is an efficient matching. To simplify notation, define $\omega_i = \mathbf{x}(\theta_i)$, $\forall i$. Then $\mathbf{y}^{\mathbf{x}}$ defines a solution to the primal form of the *assignment problem*:

$$\max_{\{\pi_{ij}\}} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} v\left(\theta_i, \omega_j\right)$$

$$\text{s.t.} \sum_{i=1}^{n} \pi_{ij} \leqslant 1, \sum_{j=1}^{n} \pi_{ij} \leqslant 1, \pi_{ij} \geqslant 0, \forall i, j.$$

The corresponding dual program is:

$$\min_{u_i, \mathbf{p}} \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} \mathbf{p}\left(\omega_j\right)$$

$$\text{s.t. } u_i + \mathbf{p}\left(\omega_j\right) \geqslant v\left(\theta_i, \omega_j\right), \forall i, j.$$

By the strong duality theorem of linear programming, the primal program has a solution if and only if the dual program does. Moreover, the dual constraints bind at the optimal primal allocation $\mathbf{y}^{\mathbf{x}}$. Hence, letting $\mathbf{p}$, $\mathbf{u}$ denote solutions to the dual program,

$$\exists\, \mathbf{p}, \mathbf{u} \text{ s.t. } u_i = v\left(\theta_i, \mathbf{x}\left(\theta_i\right)\right) - \mathbf{p}\left(\mathbf{x}\left(\theta_i\right)\right),\ u_i \geqslant v\left(\theta_i, \omega_j\right) - \mathbf{p}\left(\omega_j\right), \forall i, j.$$

Thus, under prices $\mathbf{p}$, each consumer i prefers her commodity $\mathbf{x}(\theta_i)$ to all other goods ω_j, and (WE1) is satisfied. Finally, if vectors $\mathbf{p}$ and $\mathbf{u}$ constitute a solution to the dual program, vectors $\tilde{\mathbf{p}}(\omega) = \mathbf{p}(\omega) + c$ and $\tilde{\mathbf{u}} = \mathbf{u} - c$ also form a solution for any scalar c. Thus, given solutions $\mathbf{p}$ and $\mathbf{u}$ to the dual program, let $\tilde{\mathbf{p}} = \mathbf{p} + \min_i u_i$, $\tilde{\mathbf{u}} = \mathbf{u} - \min_i u_i$. By construction,

$$\tilde{u}_i = v\left(\theta_i, \mathbf{x}\left(\theta_i\right)\right) - \tilde{\mathbf{p}}\left(\mathbf{x}\left(\theta_i\right)\right) \geqslant 0, \forall i.$$

Hence, all consumers prefer their commodity to the empty set, and (WE2) is also satisfied. Hence $\mathbf{x}$ defines a Walrasian equilibrium allocation under prices $\tilde{\mathbf{p}}$. □

B Proof of Corollary 1

Proof. By Theorem 1, it is enough to characterize the set of pairs $(\mathbf{y}, \mathbf{p})$ such that the Walrasian equilibrium conditions (WE1) and (WE2) hold for the economy with endowment C. Building on our proof of Lemma 2 in Appendix A, by the strong duality theorem of linear programming, the inequalities

$$v(\theta_i, \mathbf{x}(\theta_i)) - \mathbf{p}(\mathbf{x}(\theta_i)) \geqslant v(\theta_i, \omega_j) - \mathbf{p}(\omega_j), \forall i, j,$$

are satisfied for any primal solution $\mathbf{y}^x$ and any dual solution $\mathbf{p}$. Hence, the set of Walrasian equilibrium pairs is a product set $\mathbf{Y} \times \mathbf{P}$, where $\mathbf{Y}$ is the set of efficient allocations.

To characterize $\mathbf{P}$, suppose functions $\mathbf{p}$, $\mathbf{p}'$ satisfy (WE1) and (WE2) for some $\mathbf{x}$. Then, the function $\bar{\mathbf{p}}$ defined by $\bar{\mathbf{p}}(\omega) = \max(\mathbf{p}(\omega), \mathbf{p}'(\omega))$, the function $\underline{\mathbf{p}}$ defined by $\underline{\mathbf{p}}(\omega) = \min(\mathbf{p}(\omega), \mathbf{p}'(\omega))$ and also the function $\tilde{\mathbf{p}}(\omega) = a\mathbf{p}(\omega) + (1-a)\mathbf{p}'(\omega)$, $0 \leqslant a \leqslant 1$ also satisfy (WE1) and (WE2), by direct inspection. Moreover, if $\mathbf{p}_1$, $\mathbf{p}_2, \ldots$ satisfy (WE1) and (WE2) for all n, and $\mathbf{p}_n \to \mathbf{p}$ pointwise, then $\mathbf{p}$ also satisfies (WE1) and (WE2). Hence $\mathbf{P}$ is a convex complete sublattice of $\mathbb{R}^n$.

To see that $\mathbf{P}$ is bounded from above, note that for a economy with goods space C, $v(\theta, x)$ takes on a total of n^2 finite values, hence is bounded above by some constant M. Hence prices $\mathbf{p}(\omega) = M$, $\forall \omega$, form an upper bound for the price functions. □

C Extension to Arbitrary Type Spaces

We assume that Θ is an arbitrary space, potentially infinite. We assume that the utility function v is uniformly bounded from below, i.e. there exists $\underline{v} \in \mathbb{R}$ such that $\forall \theta \in \Theta$, $\forall x \in \mathcal{X}$, $v(\theta, x) \geqslant \underline{v}$.

Theorem 2. *The allocation rule $\mathbf{x} : \Theta \to \mathcal{X}$ is implementable if and only if for any finite $\Theta_0 \subset \Theta$, the matching defined by $\mathbf{x}|_{\Theta_0}$ is efficient.*

Proof. The necessity of the condition is obvious. We prove sufficiency. For every finite Θ_0, let $\bar{\mathbf{p}}(\mathbf{x}(\theta); \Theta_0)$ be the componentwise highest Walrasian equilibrium price for $\mathbf{x}(\theta)$ in the economy defined by $\mathbf{x}|_{\Theta_0}$. From our result in Corollary 1, $\bar{\mathbf{p}}(\mathbf{x}(\theta);\Theta_0)$ is well-defined for any $\mathbf{x}(\theta)$, Θ_0. Define transfers, for every $\theta \in \Theta$, by

$$\mathbf{t}(\theta) = \inf\{\bar{\mathbf{p}}(\mathbf{x}(\theta), \Theta_0) : |\Theta_0| < \aleph_0\}.$$

Note that $\mathbf{t}(\theta)$ is well defined because, for every Θ_0, under maximum equilibrium prices $\bar{\mathbf{p}}(\mathbf{x}(\cdot);\Theta_0)$, there exists a type $\theta^\star \in \Theta_0$ who gets zero utility in equilibrium. Indeed, if such a type did not exist, we could raise all prices by some small ϵ, contradicting the fact that we had highest equilibrium prices. Consequently $\bar{\mathbf{p}}(\mathbf{x}(\theta);\Theta_0) \geqslant v(\theta^\star, \mathbf{x}(\theta)) \geqslant \underline{v}$, and so $\mathbf{t}(\theta)$ is bounded from below by $\underline{v}$.

It is immediate that condition (IR) holds for type θ. Fixing a type $\hat{\theta}$, we will show that the condition (IC) holds as well. Fix $\epsilon > 0$. By definition, we can find a finite Θ_0 such that $\bar{\mathbf{p}}(\mathbf{x}(\hat{\theta});\Theta_0) \leqslant \mathbf{t}(\hat{\theta}) + \epsilon$. Suppose that $\theta \in \Theta_0$. Then,

$$v(\theta, \mathbf{x}(\theta)) - \mathbf{t}(\theta) \overset{(1)}{\geqslant} v(\theta, \mathbf{x}(\theta)) - \bar{\mathbf{p}}(\mathbf{x}(\theta);\Theta_0) \overset{(2)}{\geqslant} \\ v(\theta, x(\hat{\theta})) - \bar{\mathbf{p}}(\mathbf{x}(\hat{\theta});\Theta_0) \overset{(3)}{\geqslant} v(\theta, \mathbf{x}(\hat{\theta})) - \mathbf{t}(\hat{\theta}) - \epsilon, \quad \text{(C.1)}$$

where (1) follows from the definition of $\mathbf{t}(\theta)$, (2) is a Walrasian equilibrium condition, and (3) is true by the choice of Θ_0. Because ϵ was arbitrary, the inequality is proven in this case. If $\theta \notin \Theta_0$, then we instead consider $\Theta_0' := \Theta_0 \cup \{\theta\}$. Note that if $\mathbf{p} : \Theta_0' \to \mathbb{R}$ is an equilibrium price vector for the economy defined by $\mathbf{x}|_{\Theta_0'}$, then the truncation of $\mathbf{p}$ to Θ_0 must be an equilibrium price vector for the economy defined by $\mathbf{x}|_{\Theta_0}$. This means that the set of equilibrium prices for $\mathbf{x}(\hat{\theta})$ shrinks as we move from Θ_0 to Θ_0', so the maximum can only decrease:

$$\bar{\mathbf{p}}(\mathbf{x}(\hat{\theta});\Theta_0') \leqslant \bar{\mathbf{p}}(\mathbf{x}(\hat{\theta});\Theta_0).$$

We can now apply the same reasoning as before to Θ_0'. □

Remark. The condition that v is bounded from below can be relaxed if we drop the (IR) constraint. We can then make sure that $\mathbf{t}$ is well defined by fixing the utility (equivalently, the transfer) of an arbitrary type in Θ.